Asymptotic Expansion of the Quadratic Discriminant Function
for Large Dimension and Samples

Chieko MATSUMOTO and Hirofumi WAKAKI

ABSTRACT. For the classification problem between two normal populations with different covariance matrix, we consider the quadratic discriminant function (QDF shortly) and the sample quadratic discriminant function (SQDF shortly). Our purpose of this paper is to derive an asymptotic expansion of misclassification probabilities for QDF and SQDF when both the sample sizes and the dimension are large.

1. Introduction

Consider the problem of classifying an observation vector $x$ into one of two normal populations $\Pi_1: N_p(\mu_1, \Sigma_1)$ and $\Pi_2: N_p(\mu_2, \Sigma_2)$. If the mean vector $\mu_i$ and the covariance matrix $\Sigma_i$ are unknown, suppose that we have a training sample of $x_{11}, \cdots, x_{1N_1}$ from $\Pi_1$ and another independent training sample of $x_{21}, \cdots, x_{2N_2}$ from $\Pi_2$. Let $\bar{x}_i$ and $S_i$ $(i = 1, 2)$ be the sample mean and the sample covariance matrix given by

$$\bar{x}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} x_{ij}, \quad S_i = \frac{1}{n_i} \sum_{j=1}^{N_i} (x_{ij} - \bar{x}_i)(x_{ij} - \bar{x}_i)^{\prime},$$

respectively, where $n_i = N_i - 1$.
We define a discriminant function by

\[ Q(x; \mu_1, \mu_2, \Sigma_1, \Sigma_2) = (x - \mu_1)' \Sigma_1^{-1} (x - \mu_1) - (x - \mu_2)' \Sigma_2^{-1} (x - \mu_2) + \log |\Sigma_1 \Sigma_2^{-1}|, \tag{1.1} \]

if the parameters \( \mu_i \) and \( \Sigma_i \) are known. When the parameters are unknown, we define a discriminant function by

\[ Q(x; \bar{x}_1, \bar{x}_2, S_1, S_2) = (x - \bar{x}_1)' S_1^{-1} (x - \bar{x}_1) - (x - \bar{x}_2)' S_2^{-1} (x - \bar{x}_2) + \log |S_1 S_2^{-1}|, \tag{1.2} \]

using \( \bar{x}_i \) and \( S_i \) replaced by \( \mu_i \) and \( \Sigma_i \) in (1.1). We usually call (1.1) as the quadratic discriminant function (QDF shortly) and (1.2) as the sample quadratic discriminant function (SQDF shortly), respectively.

For the most part, the exact distributions of discriminant functions are too complicated to handle. Then, we often use asymptotic approximations instead of the exact distributions.

If we consider \( \Sigma_1 = \Sigma_2 \), many results about asymptotic approximations were obtained. (For the discriminant rules when \( \Sigma_1 = \Sigma_2 \), for example, W-rule and Z-rule, see Siotani, Hayakawa and Fujikoshi (1985), e.g.) When only the sample sizes are large, Okamoto (1963,1968) derived an asymptotic expansion formula for the W-rule. For the Z-rule, Memon and Okamoto (1971) derived an asymptotic expansion formula. It is known that the accuracy of these formulas depends on the dimension and the Mahalanobis distance between the populations. Therefore, when the dimension is large, the approximations by these expansion formulas are poor.

Recently Saranadasa (1993) obtained the limiting distribution function of the Z-rule when both the sample sizes and the dimension are large. Fujikoshi and Seo (1998) also derived a limiting distribution in the class of discriminant function included both the W-rule and the Z-rule, and their
numerical experiments show that the asymptotic expansion formula gives a
good approximation even for the small dimension. Moreover, Tonda and
Wakaki (2003) derived an asymptotic expansion of misclassification proba-
bility for the W-rule. Matsumoto (2004) derived an asymptotic expansion
of misclassification probability in the class included the the W-rule and the
Z-rule.

In the case of proportional covariance matrices, Wakaki (1990) derived
an asymptotic expansion of the SQDF when only the sample sizes are large.
However, when there are no restrictions about the covariance matrices, we
cannot find any asymptotic expansion formulas of the distribution function
for the QDF and the SQDF. One of the reasons is the complexity of the
limiting distribution of the SQDF, which is distributed as the weight sum of
independent noncentral chi-square distribution. When both the sample sizes
and the dimension are large, we can apply the central limiting theorem for the
weight sum. Our purpose of this paper is to derive an asymptotic expansion
of the distribution function of the SQDF when both the sample sizes and
the dimension are large. In Section 2 we derive the asymptotic expansions
of misclassification probability for the QDF and the SQDF. In Section 3,
some numerical experiments are carried out to examine the performance of
the derived asymptotic approximations and to compare with the previous
results. And for the Appendix we show the parameters of derived result.

2. Asymptotic expansion

We may derive the asymptotic approximations of the misclassification
probability of the QDF and the SQDF using the Edgeworth expansion for
large dimension and sample sizes. Then we assume that

\[ \lim_{n \to \infty} \inf \frac{p}{n_k} > 0, \quad n_k > p, \quad (2.1) \]
and define
\[ \nu_i = \frac{p}{n_i}. \]  
when the sample size \( N_i \) and the dimension \( p \) are sufficiently large.

2.1. Asymptotic expansion of the misclassification probability

The following sections, we assume that an observation vector \( \mathbf{x} \) comes from \( \Pi_1 \).

We consider the formal asymptotic expansion of the distribution of the sum of the random variables which are independent but aren’t distributed identically. Let \( X_1, X_2, \ldots, X_n \) be mutually independent random variables having \( \mathbb{E}(X_j) = 0 \) and \( \mathbb{E}(|X_j|^k) < \infty \) for each \( j \), with integral \( k \geq 3 \), and let \( \mathbb{E}(X_j^2) = \sigma_j^2 \) and \( \sum_{j=1}^n \sigma_j^2 > 0 \). Let \( F_n \) be the distribution function of \( \sum_{j=1}^n X_j / \sum_{j=1}^n \sigma_j^2 \). Then, the approximation of \( F_n(x) \) is given by

\[
F_n(x) \sim \Phi(x) - \phi(x) \left\{ \frac{(x^2 - 1)\kappa_3}{6\sqrt{\kappa^3}} + \frac{(x^3 - 3x)\kappa_4}{24\nu^2} + \frac{(x^5 - 10x^3 + 15x)\kappa_5}{72\nu^3} \right\} + o(n^{-1}),
\]

where \( \nu \) is the sum of the variance and \( \kappa_i \) is the sum of the \( i \)-th cumulant of \( X_j \). As later mentioned, the QDF and the SQDF are shown as the sum of (conditional) independent random vectors. Hence, we use (2.3) to calculate the asymptotic expansions of the misclassification probability of the QDF and the SQDF.

For the validity of (2.3), we show the following theorem.

**Theorem 2.1.** (Bhattacharya and Rao(1976)). Let \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots \) be a sequence of independent random vectors in \( \mathbb{R}^k \) with zero means and positive definite covariance matrices \( \mathbf{V}[\mathbf{x}_j] \). Assume that
\[ \mathbb{E}[|x_j|^s] < \infty \quad (\text{for some } s > 3); \]
\[ \lim_{n \to \infty} \inf \frac{1}{n} B_n^2 > 0 \quad B_n^2 = \sum_{j=1}^{n} V[x_j]; \]
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \int_{|x_j| > \epsilon \sqrt{n}} |x_j|^p d(P_{x_j}) = 0 \quad \text{for any } \epsilon > 0; \quad \text{and} \]
\[ g_n(t) \text{: the characteristic function of } x_n, \quad \lim_{n \to \infty} \sup_{|t| > \delta} |g_n(t)| < 1 \quad \text{for any } \delta > 0. \]

In this time, the distribution function of \( B_n^{-1} \sum_{j=1}^{n} x_j \) can product Edgeworth expansion under \( O(n^{-(s-2)/2}). \) (The formal expansion with cumulant has the validity.)

However, when the sample sizes and the dimension are large, we have to improve Theorem 2.1. Nevertheless, for a little improvement, we’ll check the validity of (2.3) for the QDF and the SQDF. If (2.3) has the validity, the misclassification probability for the QDF and the SQDF is asymptotically given by

\[
\Pr(2|1) = P[Q(x) > -2 \log c|H_1] \\
\sim \Phi(-a) + \phi(a) \left\{ (a^2 - 1) \frac{\kappa_3[Q(x)]}{6 \Var[Q(x)]^{3/2}} + (a^3 - 3a) \frac{\kappa_4[Q(x)]}{24 \Var[Q(x)]^2} \right\} + o(p^{-1}),
\]

using (2.3), where \( Q(x) \) is shown as the QDF or the SQDF,

\[ a = \frac{-2 \log c - \mathbb{E}[Q(x)]}{\Var[Q(x)]^{1/2}}, \]

and \( \kappa_i[Q(x)] \) is the \( i \)-th cumulant of \( Q(x) \). Therefore, we have to calculate the expectation, the variance, the 3rd and the 4th cumulant of the QDF and the SQDF. For applying these results to (2.4), we may obtain the asymptotic expansions of misclassification probability.
2.2. Asymptotic expansion of misclassification probability for the quadratic discriminant function

First, we consider the asymptotic expansion of misclassification probability for the QDF (1.1) up to $p^{-1/2}$. In this case, we assume that the means and the covariance matrices are known. For making the discussion simply, we standardize $x$ and $x_{ij}$ as

\[ y = L(x - \mu_1), \quad y_{ij} = L(x_{ij} - \mu_1) \quad (i = 1, 2; j = 1, \cdots, N_i), \quad (2.5) \]

where $L$ is the nonsingular matrix such that $L^T \Sigma_1 L = I_p$ and $L^T \Sigma_2 L = L = \text{diag}(\lambda_1, \cdots, \lambda_p)$, with $\lambda_i$’s being the roots of $|\Sigma_2 - \lambda \Sigma_1| = 0$. Then, the quadratic discriminant function (1.1) is expressed as

\[
Q(y; 0, \eta, I_p, \Lambda_p) = y^T I_p y - (y - \eta)^T \Lambda_p^{-1} (y - \eta) + \sum_{i=1}^{p} \log \lambda_i^{-1} = \sum_{j=1}^{p} \{y_j^2 - \lambda_j^{-1} (y_j - \eta_j)^2 + \log \lambda_j^{-1} \},
\]

(2.6) is the form of the sum of the independent random vectors. Then, the expectation, variance and 3rd cumulant of statistic $T = y^T (y - \eta)^T \Lambda_p^{-1} (y - \eta)$ are calculated respectively as

\[
\begin{align*}
E[T] &= p - (\text{tr} \Lambda_p^{-1} + \eta^T \Lambda_p^{-1} \eta), \\
\text{Var}[T] &= 2p - 4\text{tr} \Lambda_p^{-1} + 2\text{tr} \Lambda_p^{-2} + 4\eta^T \Lambda_p^{-2} \eta, \\
\kappa_3[T] &= 8p + 24\eta^T (\Lambda_p^{-2} - \Lambda_p^{-3}) \eta - 24\text{tr} \Lambda_p^{-1} + 24\text{tr} \Lambda_p^{-2} - 8\text{tr} \Lambda_p^{-3}.
\end{align*}
\]

For the validity, we consider the following lemma. (About the proof, see Theorem 2.1.)
**Lemma 2.1.** Denote that \( \epsilon_0 := E[Q(y; 0, \eta, I_p, \Lambda_p)] \), \( \nu_0 := \text{Var}[Q(y; 0, \eta, I_p, \Lambda_p)] \) and \( \kappa := \kappa_3[Q(y; 0, \eta, I_p, \Lambda_p)] \). If
\[
\lim_{p \to \infty} \frac{1}{p} \left( p - 2\text{tr}\Lambda_p^{-1} + \text{tr}\Lambda_p^{-2} + 2\eta / \Lambda_p^{-2}\eta \right) > 0,
\]
\[
\frac{1}{p} \sum_{j=1}^{p} \lambda_j^3 |\eta_j|^k = O(1) \quad k = 0, 1, \ldots, 6,
\]
then uniformly for \( x \),
\[
\mathbb{P}(\frac{Q(y; 0, \eta, I_p, \Lambda_p) - \epsilon_0}{\nu_0^{1/2}} \leq x) = \Phi(x) + \phi(x)(x^2 - 1)\frac{\kappa}{6\nu_0^{3/2}} + o(p^{-1/2}).
\]

Then, we obtain the following theorem.

**Theorem 2.2.** The misclassification probability of quadratic discriminant function (1,1) is asymptotically given as
\[
\mathbb{P}(\mathbf{2|1}) \sim \Phi(-a) + \phi(a)(a^2 - 1)\frac{\kappa}{6\nu_0^{3/2}} + o(\frac{1}{\sqrt{p}}),
\]
(2.7)
where
\[
a = \frac{-2\log c - \epsilon_0}{\nu_0^{1/2}},
\]
\[
\epsilon_0 = p - (\text{tr}\Lambda_p^{-1} + \eta / \Lambda_p^{-1}\eta) + \sum_{j=1}^{p} \log \lambda_j^{-1},
\]
\[
\nu_0 = 2p - 4\text{tr}\Lambda_p^{-1} + 2\text{tr}\Lambda_p^{-2} + 4\eta / \Lambda_p^{-2}\eta,
\]
\[
\kappa = 8p + 24\eta / (\Lambda_p^{-2} - \Lambda_p^{-3})\eta - 24\text{tr}\Lambda_p^{-1} + 24\text{tr}\Lambda_p^{-2} - 8\text{tr}\Lambda_p^{-3}.
\]

When we assume that \( a = O(1) \), i.e.
\[
p - (\text{tr}\Lambda_p^{-1} + \eta / \Lambda_p^{-1}\eta) = O(\sqrt{p}) \quad (p \to \infty),
\]
(2.8)
the approximation will be good.
2.3. Asymptotic expansion of misclassification probability for the sample quadratic discriminant function

Next, we consider the asymptotic expansion of misclassification probability for the SQDF (1.2). In the same way for the QDF, we standardize $\mathbf{X}$ and $\mathbf{X}_{ij}$ using (2.5). Then, we obtain

$$Q(y; \bar{y}_1, \bar{y}_2, B_1, B_2) = n_1(y - \bar{y}_1)^T B_1^{-1}(y - \bar{y}_1) - n_2(y - \bar{y}_2)^T B_2^{-1}(y - \bar{y}_2) + \log |B_1/B_2| - \rho \log(n_1/n_2),$$  \hspace{1cm} (2.9)

where $n_i B_i = L S_i L$ and the random variables $y, y_{ij}, y_{xj}, \bar{y}_1, \bar{y}_2, B_1$ and $B_2$ are distributed as

$$y, y_{ij} \sim N_p(0, I_p), \quad y_{xj} \sim N_p(\eta, \Sigma_p), \quad \eta = L(\mu_2 - \mu_1),$$

$$\bar{y}_1 \sim N_p(0, \frac{1}{N_1} I_p), \quad \bar{y}_2 \sim N_p(\eta, \frac{1}{N_2} \Sigma_p),$$

$$B_1 \sim W_p(n_1, I_p), \quad B_2 \sim W_p(n_2, \Sigma_p),$$

respectively. Besides, we need to show (2.9) as the independent random variables to calculate the asymptotic approximation of misclassification probability using (2.4). For using some theorems (see Muirhead (1982), e.g.), we obtain the following result.

**Lemma 2.2.** The sample quadratic discriminant function (2.9) is represented as

$$Q(y; \bar{y}_1, \bar{y}_2, B_1, B_2)$$

$$= n_1 V_{11}^{-1}(y - \bar{y}_1)^T(y - \bar{y}_1) - n_2 V_{21}^{-1}(y - \bar{y}_2)^T \lambda_p^{-1}(y - \bar{y}_2)$$

$$+ \sum_{k=1}^{p} (\log(V_{1k}/n_1) - \log(V_{2k}/n_2) - \log \lambda_k)$$

$$= n_1 V_{11}^{-1} \sum_{i=1}^{p} (y_i - \bar{y}_1)^2 - n_2 V_{21}^{-1} \sum_{i=1}^{p} \lambda_i^{-1} (y_i - \bar{y}_2)^2$$

$$+ \sum_{k=1}^{p} (\log(V_{1k}/n_1) - \log(V_{2k}/n_2) - \log \lambda_k),$$ \hspace{1cm} (2.10)
where \( \mathbf{y} = (y_1, \cdots, y_p)' \), \( \mathbf{\tilde{y}}_1 = (\mathbf{\tilde{y}}_{11}, \cdots, \mathbf{\tilde{y}}_{1p})' \), \( \mathbf{\tilde{y}}_2 = (\mathbf{\tilde{y}}_{21}, \cdots, \mathbf{\tilde{y}}_{2p})' \) and \( V_{ij} \) are independent and these random vectors are distributed as

\[
\mathbf{y} \sim N_p(\mathbf{0}, I_p), \quad \mathbf{\tilde{y}}_1 \sim N_p(\mathbf{0}, \frac{1}{N_1} I_p), \quad \mathbf{\tilde{y}}_2 \sim N_p(\eta, \frac{1}{N_2} \Lambda_p),
\]

\[
V_{ik} \sim \chi^2_{n_i-p+k}(i = 1, 2; k = 1, \cdots, N_i),
\]

respectively.

If \( V_{11} \) and \( V_{21} \) are given, (2.10) are rated as the sum of independent random variables. We divide (2.10) into two parts, \( Q(\mathbf{y}; \mathbf{\tilde{y}}_1, \mathbf{\tilde{y}}_2, B_1, B_2) = Q_1 + Q_2 \), where

\[
Q_1 = n_1 V_{11}^{-1}(\mathbf{y} - \mathbf{\tilde{y}}_1)'(\mathbf{y} - \mathbf{\tilde{y}}_1) - n_2 V_{21}^{-1}(\mathbf{y} - \mathbf{\tilde{y}}_2)' \Lambda_p^{-1}(\mathbf{y} - \mathbf{\tilde{y}}_2),
\]

\[
Q_2 = \sum_{k=1}^{p} (\log(V_{1k}/n_1) - \log(V_{2k}/n_2) - \log \lambda_k).
\]

For \( Q_1 \), it is easy to calculate the conditional cumulant, because \( \mathbf{y} \), \( \mathbf{\tilde{y}}_1 \) and \( \mathbf{\tilde{y}}_2 \) is distributed as normal distribution. Then, conditional expectation, variance, the 3rd and the 4th cumulant of \( Q_1 \) are shown as

\[
E[Q_1|V_1, V_2] = p \left( a_{11} \frac{n_1}{V_1} + a_{12} \frac{n_2}{V_2} \right) + \left( a_{21} \frac{n_1}{V_1} + a_{22} \frac{n_2}{V_2} \right),
\]

\[
\text{Var}[Q_1|V_1, V_2] = 2p \left( b_{11} \frac{n_1^2}{V_1^2} + b_{12} \frac{n_1 n_2}{V_1 V_2} + b_{13} \frac{n_2^2}{V_2^2} \right)
\]

\[
+ \left( b_{21} \frac{n_1^2}{V_1} + b_{22} \frac{n_2^2}{V_2^2} \right) + O(p^{-1}),
\]

\[
\kappa_3[Q_1|V_1, V_2] = 8p \left( c_{11} \frac{n_1^3}{V_1^3} + c_{12} \frac{n_1 n_2^2}{V_1^2 V_2} + c_{13} \frac{n_1^2 n_2}{V_1 V_2^2} + c_{14} \frac{n_2^3}{V_2^3} \right)
\]

\[
+ O(1),
\]

\[
\kappa_4[Q_1|V_1, V_2] = p \left( d_{11} \frac{n_1^4}{V_1^4} + d_{12} \frac{n_1^3 n_2}{V_1^3 V_2} + d_{13} \frac{n_1^2 n_2^2}{V_1^2 V_2^2} + d_{14} \frac{n_1 n_2^3}{V_1 V_2^3} + d_{15} \frac{n_2^4}{V_2^4} \right)
\]

\[
+ O(1),
\]

where

\[
a_{11} = 1, \quad a_{12} = -\frac{1}{p} (\text{tr} \Lambda^{-1} + \eta' \Lambda^{-1} \eta), \quad a_{21} = \frac{p}{N_1}, \quad a_{22} = -\frac{p}{N_2},
\]

\[9\]
\[ b_{11} = 1, \quad b_{12} = -2p^{-1}\text{tr}\Lambda^{-1}, \quad b_{13} = p^{-1}(\text{tr}\Lambda^{-2} + 2p/\Lambda^{-2} \eta), \]
\[
   c_{11} = 1, \quad c_{12} = -3p^{-1}\text{tr}\Lambda^{-1},
\]
\[
   c_{13} = 3p^{-1}(\text{tr}\Lambda^{-2} + \eta/\Lambda^{-2} \eta), \quad c_{14} = -p^{-1}(\text{tr}\Lambda^{-3} + \eta/\Lambda^{-3} \eta),
\]
\[
   d_{11} = 48, \quad d_{12} = -192p^{-1}\text{tr}\Lambda^{-1}, \quad d_{13} = 96p^{-1}(3\text{tr}\Lambda^{-2} + 2p/\Lambda^{-2} \eta), \quad d_{14} = -192p^{-1}(\text{tr}\Lambda^{-3} + 2p/\Lambda^{-3} \eta), \quad d_{15} = 48p^{-1}(\text{tr}\Lambda^{-4} + 4p/\Lambda^{-4} \eta).
\]

For \( Q_2 \), it is difficult to calculate the cumulant of \( \log(V_{ik}/n_i) \). Then, we standardize \( V_{ik} \) as
\[
   U_{ik} = \frac{V_{ik} - (n_i - p + k)}{\sqrt{2(n_i - p + k)}}.
\]

\( U_{ik} \) is asymptotically distributed as standard normal distribution. Then the expectation, variance, the 3rd and 4th cumulant of \( Q_2 \) are shown as
\[
   \text{E}(Q_2) = p\alpha_{13} + a_{23} + O(p^{-1}), \quad \text{Var}(Q_2) = b_{23} + O(p^{-1}),
   \quad \kappa_3(Q_2) \sim O(p^{-1}), \quad \kappa_4(Q_2) \sim O(p^{-2}),
\]

where
\[
   \alpha_{13} = -\frac{n_1 - p}{p} \log\left(\frac{n_1 - p}{n_1}\right) + \frac{n_2 - p}{p} \log\left(\frac{n_2 - p}{n_2}\right) - p^{-1} \sum_{k=1}^{p} \log \lambda_k,
\]
\[
   a_{23} = -\frac{1}{2} \log\left(\frac{n_1 - p}{n_1}\right) + \frac{1}{2} \log\left(\frac{n_2 - p}{n_2}\right) - \sum_{k=1}^{p} \frac{1}{n_1 - p + k} + \sum_{k=1}^{p} \frac{1}{n_2 - p + k},
\]
\[
   b_{23} = \sum_{k=1}^{p} \frac{2}{n_1 - p + k} + \sum_{k=1}^{p} \frac{2}{n_2 - p + k}.
\]
(We may need to consider the conditional independence.)

For the validity, we obtain the following lemma. (About the proof, see Theorem 2.1.)

**Lemma 2.3.** Let \( E_p = E[Q(\mathbf{y}; \tilde{y}_1, \tilde{y}_2, B_1, B_2)] \),
\[
   v_p = V[Q(\mathbf{y}; \tilde{y}_1, \tilde{y}_2, B_1, B_2)], \quad \kappa_3 = \kappa_3[Q(\mathbf{y}; \tilde{y}_1, \tilde{y}_2, B_1, B_2)] \text{ and}
\]

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\[ \kappa_4 = \kappa_4[Q(y; \bar{y}_1, \bar{y}_2, B_1, B_2)], \text{ if} \]
\[ \liminf_{p \to \infty} \frac{1}{p} \left[ \text{tr}\{(I_p - \Lambda_p^{-1})^2\} + 2\eta' \Lambda_p^{-1} \eta\right] = O(1), \]
\[ \frac{1}{p} \sum_{j=1}^{p} \lambda_j^{-k} |V_j|^{2l} = O(1) \quad (k = 1, \ldots, 3, \ l = 0, 1, \ldots, 3). \]

Then uniformly for \( x \),
\[ \Pr\left(\frac{Q(y; \bar{y}_1, \bar{y}_2, B_1, B_2)}{v_p^{1/2}} - E_p \leq x\right) = \Phi(x) + \phi(x) \left\{ (x^2 - 1) \frac{\kappa_3}{6v_p^{3/2}} + (x^3 - 3x) \frac{\kappa_4}{24v_p^2} + (x^5 - 10x^3 + 15x) \frac{\kappa_3^2}{72v_p^3} \right\} + o(p^{-1}). \]

Then, we obtain the following result.

**Theorem 2.3.** The misclassification probability of the sample quadratic discriminant function is asymptotically given by the expectation form for \( V_{11} \) and \( V_{21} \), as
\[ P(2|1) \sim E_{V_{11}, V_{21}} \left[ \Phi(-a) + \phi(a) \left\{ (a^2 - 1) \frac{\kappa_3}{6v_p^{3/2}} + (a^3 - 3a) \frac{\kappa_4}{24v_p^2} + (a^5 - 10a^3 + 15a) \frac{\kappa_3^2}{72v_p^3} \right\} \right] + o(p^{-1}), \quad (2.17) \]
where
\[ a = \frac{2 \log c + E_p}{v_p^{1/2}}, \]
\[ E_p \sim p \left( a_{11} \frac{n_1}{V_1} + a_{12} \frac{n_2}{V_2} + a_{13} \right) + \left( a_{21} \frac{n_1}{V_1} + a_{22} \frac{n_2}{V_2} + a_{23} \right) + O(p^{-1}), \quad (2.18) \]
\[ v_p \sim 2p \left( b_{11} \frac{n_1^2}{V_1^2} + b_{12} \frac{n_1 n_2}{V_1 V_2} + b_{13} \frac{n_2^2}{V_2^2} \right) \quad (2.19) \]
\[ + \left( b_{21} \frac{n_1^2}{V_1^2} + b_{22} \frac{n_1 n_2}{V_1 V_2} + b_{23} \right) + O(p^{-1}), \quad (2.20) \]
\[ \kappa_3 \sim 8p \left( c_{11} \frac{n_1^3}{V_1^3} + c_{12} \frac{n_1^2 n_2}{V_1^2 V_2} + c_{13} \frac{n_1 n_2^2}{V_1 V_2^2} + c_{14} \frac{n_2^3}{V_2^3} \right) + O(1), \quad (2.21) \]

\[ \kappa_4 \sim p \left( d_{11} \frac{n_1^4}{V_1^4} + d_{12} \frac{n_1^3 n_2}{V_1^3 V_2} + d_{13} \frac{n_1^2 n_2^2}{V_1^2 V_2^2} + d_{14} \frac{n_1 n_2^3}{V_1 V_2^3} + d_{15} \frac{n_2^4}{V_2^4} \right) + O(1). \quad (2.22) \]

To calculate the expectation of (2.17) for \( V_{11} \) and \( V_{21} \), we obtain the asymptotic expansion of the misclassification probability of the quadratic discriminant function. We use the standardization of \( V_{i1} \),

\[ U_i = \frac{V_{i1} - (n_i - p + 1)}{\sqrt{2(n_i - p + 1)}}. \quad (2.23) \]

for the calculation. Then we obtain the following result.

**Theorem 2.4.** The misclassification probability of the sample quadratic discriminant function (1.2) is asymptotically given by

\[ P(2|1) = f_1 + \frac{1}{\sqrt{p}} f_2 + \frac{1}{p} f_3 + o(p^{-1}), \quad (2.24) \]

where \( f_1, f_2 \) and \( f_3 \) are shown in Appendix.

3. Simulation

We carry out some numerical experiments for three purposes. The first purpose is to examine the accuracy of the asymptotic expansion formula given by Theorem 2.2, at the case of parameter known. The second purpose is to examine the accuracy of the asymptotic expansion formula given by Theorem 2.4, at the case of parameter unknown. And the third purpose is to compare the result in Wakaki (1990), at the case of only large samples.

3.1. Methods of the examination

We denote \( \Pi_1 \sim N_p(0, I_p) \) and \( \Pi_2 \sim N_p(\eta, \Lambda_p) \) \( (\Lambda_p = \text{diag}(\lambda_1, \cdots, \lambda_p)) \), and consider the parameters,
Table 1: The results of simulation when parameters are known.

<table>
<thead>
<tr>
<th>$\alpha$ = 1.0</th>
<th>$\zeta$</th>
<th>$p = 10$</th>
<th>$p = 30$</th>
<th>$p = 30$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Sim.</td>
<td>(2.7)</td>
<td>Sim.</td>
<td>(2.7)</td>
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<td>0.201066</td>
<td>0.200466</td>
<td>0.201066</td>
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<td>0.197071</td>
<td>0.197071</td>
<td>0.197071</td>
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<td>0.163446</td>
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</table>

$\alpha = 1.5$  
$\lambda_j = \alpha^{(p-j)/p}$  
$\lambda_j = \alpha^{(p-2j)/p}$

<table>
<thead>
<tr>
<th>$\alpha$ = 1.5</th>
<th>$\zeta$</th>
<th>$p = 10$</th>
<th>$p = 30$</th>
<th>$p = 30$</th>
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</thead>
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<td>Sim.</td>
<td>(2.7)</td>
</tr>
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<td>0.145224</td>
<td>0.145224</td>
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<td>0.081836</td>
<td>0.081836</td>
<td>0.081836</td>
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Table 2: The results of simulation when $p = 10$ and parameters are unknown.

<table>
<thead>
<tr>
<th>$p = 10$</th>
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<th>Simulation</th>
<th>First Term</th>
<th>Up to $p^{1/2}$</th>
<th>Up to $p^{-1}$</th>
</tr>
</thead>
<tbody>
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<td>\alpha = 1.0</td>
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<td>0.430105</td>
<td>0.538842</td>
<td>0.487638</td>
</tr>
<tr>
<td></td>
<td>1.65</td>
<td>0.47727</td>
<td>0.387004</td>
<td>0.518028</td>
<td>0.482358</td>
</tr>
<tr>
<td></td>
<td>1.96</td>
<td>0.44314</td>
<td>0.347019</td>
<td>0.498134</td>
<td>0.482363</td>
</tr>
</tbody>
</table>

$\alpha = 1.5$  
$\lambda_j = \alpha^{(p-j)/p}$

<table>
<thead>
<tr>
<th>$\alpha = 1.5$</th>
<th>$\zeta$</th>
<th>Simulation</th>
<th>First Term</th>
<th>Up to $p^{1/2}$</th>
<th>Up to $p^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.473576</td>
<td>0.507339</td>
<td>0.488065</td>
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</tr>
<tr>
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<td>0.52694</td>
<td>0.431938</td>
<td>0.503722</td>
<td>0.454312</td>
<td></td>
</tr>
<tr>
<td>1.96</td>
<td>0.47977</td>
<td>0.398183</td>
<td>0.491756</td>
<td>0.429851</td>
<td></td>
</tr>
</tbody>
</table>

| $\lambda_j = \alpha^{(p-2j)/p}$ |
| 1.28          | 0.44754 | 0.397475 | 0.462125 | 0.508781 |
| 1.65          | 0.41341 | 0.352555 | 0.490327 | 0.504306 |
| 1.96          | 0.38052 | 0.310994 | 0.464419 | 0.506498 |

- $p = 10, 20, 30$;
- $N_i = 1.5p$ (i = 1, 2);
- $c = 1$;
- $\lambda_j = \alpha^{(p-j)/p}$ or $\lambda_j = \alpha^{(p-2j)/p}$, where $\alpha = 1.0, 1.5$;
- $\eta = \delta(1, 2, \cdots, p)^{\gamma}$, where $\delta = \left\{ \frac{\sum_{j=1}^{p} j^2}{\frac{1}{2}} \right\}^{1/2} \times \zeta$ and $\zeta = 1.28, 1.65, 1.96$.

Then we make the samples from these parameters, and estimate the misclassification probability for (1.1) and (1.2) using Monte-Carlo methods. Besides, we calculate the asymptotic approximation by using (2.7) and (2.24). Then, we compare the result of (1.1) with that of (2.7), and the result of (1.2) with that of (2.24).
Table 3: The results of simulation when \( p = 20 \) and parameters are unknown.

<table>
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<tr>
<th>( p = 20 )</th>
<th>( \zeta )</th>
<th>Simulation</th>
<th>First Term</th>
<th>( \frac{\text{(2.24)}}{} ) Up to ( p^{-t/2} )</th>
<th>Up to ( p^{-1} )</th>
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<tr>
<td>( \alpha = 1.0 )</td>
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<td>0.40538</td>
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<td>1.65</td>
<td>0.46938</td>
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<td>0.490675</td>
<td>0.453709</td>
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<tr>
<td></td>
<td>1.96</td>
<td>0.44089</td>
<td>0.388872</td>
<td>0.472055</td>
<td>0.431448</td>
</tr>
<tr>
<td>( \alpha = 1.5 )</td>
<td>( \lambda_j = e^{(\beta-j)/p} )</td>
<td>1.28</td>
<td>0.59701</td>
<td>0.516046</td>
<td>0.491269</td>
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<tr>
<td></td>
<td>1.65</td>
<td>0.56635</td>
<td>0.483104</td>
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<td>1.96</td>
<td>0.53539</td>
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<td>0.480683</td>
<td>0.462304</td>
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<tr>
<td>( \lambda_j = e^{(\beta-2j)/p} )</td>
<td>1.28</td>
<td>0.41743</td>
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<tr>
<td></td>
<td>1.65</td>
<td>0.38696</td>
<td>0.382673</td>
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<td>1.96</td>
<td>0.35879</td>
<td>0.349117</td>
<td>0.441681</td>
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Table 4: The results of simulation when \( p = 30 \) and parameters are unknown.

<table>
<thead>
<tr>
<th>( p = 30 )</th>
<th>( \zeta )</th>
<th>Simulation</th>
<th>First Term</th>
<th>( \frac{\text{(2.24)}}{} ) Up to ( p^{-t/2} )</th>
<th>Up to ( p^{-1} )</th>
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<tr>
<td>( \alpha = 1.0 )</td>
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<td>0.49564</td>
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<tr>
<td>( \alpha = 1.5 )</td>
<td>( \lambda_j = e^{(\beta-j)/p} )</td>
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<td>0.487004</td>
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<td>0.480193</td>
</tr>
<tr>
<td>( \lambda_j = e^{(\beta-2j)/p} )</td>
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<td>0.367108</td>
<td>0.435737</td>
<td>0.417622</td>
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</table>

3.2 Results and comments

Table 1 shows the result to compare (2.7) with the numerical result using (1.1). Table 2, 3 and 4 show the results to compare (2.24) with the numerical result using (1.2) when \( p = 10, 20 \) and 30 respectively. Standard deviations of these numerical simulations are not over 0.01, everything.

In this simulation, it is shown that (2.7) has a good approximation. But (2.24) doesn’t have a good approximation much. Its reasons are, we consider, that

- The sample quadratic discriminant function is the distribution of weight sum of independent noncentral chi-square random variables;
Table 5: Comparing the results with Wakaki(1990) (1).

<table>
<thead>
<tr>
<th>$p$</th>
<th>$N$</th>
<th>$\mu$</th>
<th>$\lambda$</th>
<th>Sim.</th>
<th>Wakaki’s result</th>
<th>(224)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>First Term</td>
<td>Up to $N^{-1}$</td>
</tr>
<tr>
<td>10</td>
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<td>0.4610</td>
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<td>0.4427</td>
</tr>
<tr>
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<td>0.4025</td>
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<td>0.2821</td>
<td>0.2880</td>
</tr>
</tbody>
</table>

- The dimension $p$ of this simulation is too small; and
- As the condition for the parameters $\left( \sum_{j=1}^{p} \lambda_{j}^{-2} |\eta_{j}|^{2} \right) / p$, its size has effect on the actual error.

### 3.3 Comparison

To compare the result of Wakaki (1990), we denote $\Pi_{1} \sim N_{p}(0, I_{p})$ and $\Pi_{2} \sim N_{p}(\mu, I_{p})$, where $N_{1} = N_{2} = N$ and $\mu = (\mu_{1}, 0, \cdots, 0)^{T}$. Then we consider the parameters,

- $p = 5, 10, 20$;
Table 6: Comparing the results with Wakaki(1990) (2).

<table>
<thead>
<tr>
<th>p</th>
<th>N</th>
<th>μ</th>
<th>λ</th>
<th>Sim.</th>
<th>Wakaki's result</th>
<th>[2.24]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>First Term</td>
<td>Up to N^{-1}</td>
</tr>
<tr>
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<td>1.2</td>
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<td>0.7099</td>
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- \( N = 10, 15, 20, 30, 45, \) where \( N \geq 1.5p; \)
- \( \mu = 1.5, 2, 2.5; \)
- \( \lambda = 1.2, 1.6. \)

For the same methods in Section 3.1, we obtain the numerical results at Table 5, 6 and 7. In this simulation, it is show that the approximation in (2.24) is better than Wakaki’s works for large dimension and sample sizes. In Wakaki’s results, the dimension \( p \) is used for the parameter of \( \chi^2 \)-distribution. Therefore, the approximation by his paper will be poor for large dimension.
### Table 7: Comparing the results with Wakaki(1990) (3).

<table>
<thead>
<tr>
<th>p</th>
<th>N</th>
<th>μ</th>
<th>λ</th>
<th>Sim.</th>
<th>Wakaki’s result</th>
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<td>0.8845</td>
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<td>0.4198</td>
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### A. Appendix – Parameters

We may show the parameters of the derived results in Section 2.3. Using $U_i$ of (2.23), standardization of $V_{i1}$, (2.18), (2.20), (2.21) and (2.22) are shown as

\[
E_p = pr_0 + \sqrt{p}(r_{11}U_1 + r_{12}U_2) + (r_{21}U_1^2 + r_{22}U_2^2 + r_{23})
\]

\[
\frac{1}{\sqrt{p}}(r_{31}U_1^3 + r_{32}U_2^3 + r_{33}U_1 + r_{34}U_2) + O(p^{1}),
\]

\[
v_p = ps_0 + \sqrt{p}(s_{11}U_1 + s_{12}U_2) + (s_{21}U_1^2 + s_{22}U_1U_2 + s_{23}U_2^2 + s_{24})
\]

\[
\frac{1}{\sqrt{p}}(s_{31}U_1^3 + s_{32}U_2^3U_2 + s_{33}U_1U_2^2 + s_{34}U_1^3 + s_{35}U_2 + s_{36}U_2) + O(p^{1}),
\]

\[
\kappa_3 = pk_0 + \sqrt{p}(t_{11}U_1 + t_{12}U_2) + O(1),
\]

\[
\kappa_4 = pw_0 + O(p^{1/2}),
\]

where

\[
r_0 = \frac{n_1}{n_1 - p + 1}a_{11} + \frac{n_2}{n_2 - p + 1}a_{12} + a_{13},
\]

\[
r_{11} = \sqrt{\frac{p}{n_1 - p + 1}a_{11}}, \quad r_{12} = -\sqrt{\frac{p}{n_2 - p + 1}a_{12}},
\]

\[
r_{21} = \frac{n_1}{n_1 - p + 1}a_{11}, \quad r_{22} = \frac{n_2}{n_2 - p + 1}a_{12},
\]

\[
r_{23} = \frac{p}{n_1 - p + 1}a_{21} + \frac{p}{n_2 - p + 1}a_{22} + a_{23},
\]

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\[
\begin{align*}
    r_{31} &= -2\sqrt{2} \frac{p^{3/2}n_1}{(n_1 - p + 1)^{3/2}} a_{11}, \\
    r_{32} &= -2\sqrt{2} \frac{p^{3/2}n_2}{(n_2 - p + 1)^{3/2}} a_{12}, \\
    r_{33} &= -\sqrt{2} \frac{p^{1/2}n_1}{(n_1 - p + 1)^{3/2}} a_{21}, \\
    r_{34} &= -\sqrt{2} \frac{p^{1/2}n_2}{(n_2 - p + 1)^{3/2}} a_{22}, \\
    s_0 &= 2 \left( \frac{n_1^2}{(n_1 - p + 1)^2} b_{11} + \frac{n_1 n_2}{(n_1 - p + 1)(n_2 - p + 1)} b_{12} + \frac{n_2^2}{(n_2 - p + 1)^2} b_{13} \right), \\
    s_{11} &= -2\sqrt{2} \frac{p^{1/2}n_1}{(n_1 - p + 1)^{3/2}} \left( \frac{n_1}{n_1 - p + 1} b_{11} + \frac{n_2}{n_2 - p + 1} b_{12} \right), \\
    s_{12} &= -2\sqrt{2} \frac{p^{1/2}n_2}{(n_2 - p + 1)^{3/2}} \left( \frac{n_2}{n_2 - p + 1} b_{13} + \frac{n_1}{n_1 - p + 1} b_{12} \right), \\
    s_{21} &= 4 \frac{p n_1}{(n_1 - p + 1)^2} \left( \frac{n_1}{n_1 - p + 1} b_{11} + \frac{n_2}{n_2 - p + 1} b_{12} \right), \\
    s_{22} &= 4 \frac{p n_2}{(n_2 - p + 1)^2} \left( \frac{n_2}{n_2 - p + 1} b_{13} + \frac{n_1}{n_1 - p + 1} b_{12} \right), \\
    s_{24} &= 4 \frac{r_{31} r_{32}}{(n_1 - p + 1)^2 (n_2 - p + 1)^3/2} b_{21} + b_{23}, \\
    s_{31} &= -4\sqrt{2} \frac{p^{3/2}n_1}{(n_1 - p + 1)^{5/2}} \left( 4 \frac{n_1}{n_1 - p + 1} b_{11} + \frac{n_2}{n_2 - p + 1} b_{12} \right), \\
    s_{32} &= -4\sqrt{2} \frac{p^{3/2}n_2}{(n_2 - p + 1)^{5/2}} \left( 4 \frac{n_2}{n_2 - p + 1} b_{13} + \frac{n_1}{n_1 - p + 1} b_{12} \right), \\
    s_{33} &= -4\sqrt{2} \frac{p^{3/2}n_1 n_2}{(n_1 - p + 1)^3 (n_2 - p + 1)^2} b_{12}, \\
    s_{34} &= -4\sqrt{2} \frac{p^{3/2}n_2}{(n_2 - p + 1)^{5/2}} \left( 4 \frac{n_2}{n_2 - p + 1} b_{13} + \frac{n_1}{n_1 - p + 1} b_{12} \right), \\
    s_{35} &= -8\sqrt{2} \frac{p^{1/2}n_1^2}{(n_1 - p + 1)^{5/2}} b_{21}, \\
    s_{36} &= -8\sqrt{2} \frac{p^{1/2}n_1^2}{(n_1 - p + 1)^{5/2}} b_{22} + \frac{n_1^2 n_2}{(n_1 - p + 1)^2} b_{22}, \\
    t_0 &= 8 \left( \frac{n_1^3}{(n_1 - p + 1)^2} c_{11} + \frac{n_1^2 n_2}{(n_1 - p + 1)^2} c_{12} + \frac{n_2^3}{(n_2 - p + 1)^2} c_{14} \right), \\
    t_{11} &= -8\sqrt{2} \frac{p^{1/2}n_1^2}{(n_1 - p + 1)^{3/2}} \left( 3 \frac{n_1^2 n_2}{(n_1 - p + 1)^2} c_{11} \right),
\end{align*}
\]
\[
\begin{align*}
+2 \frac{p^{1/2} n_1 n_2}{(n_2 - p + 1)^{3/2}} c_{12} + \frac{p^{1/2} n_2^2}{(n_2 - p + 1)^2} c_{13},
\end{align*}
\]

\[
\begin{align*}
t_{12} &= -8\sqrt{2} \frac{p^{1/2} n_2}{(n_2 - p + 1)^{3/2}} \left(3 \frac{p^{1/2} n_2}{(n_2 - p + 1)^2} c_{14}
\right)
+ \frac{p^{1/2} n_1 n_2}{(n_2 - p + 1)(n_2 - p + 1)} c_{13}
+ \frac{p^{1/2} n_1^2}{(n_1 - p + 1)^2} c_{12},
\end{align*}
\]

\[
\begin{align*}
w_0 &= \left(\frac{n_1}{n_1 - p + 1}\right)^4 d_{11} + \frac{n_1 n_2}{(n_1 - p + 1)^2} d_{12}
+ \frac{n_2}{(n_2 - p + 1)^2} d_{13} + \frac{n_1 n_2}{(n_1 - p + 1)(n_1 - p + 1)} d_{14}
+ \frac{n_2^3}{(n_2 - p + 1)^4} d_{15}.
\end{align*}
\]

For the condition of the good approximation and the expansion, if we assume that \( r_0 \times \sqrt{p} = O(1) \) and \( s_0 = O(1) \), \( a, \kappa_3/6v_p^3 \), \( \kappa_4/24v_p^3 \) and \( \kappa_3/72v_p^3 \) of (2.17) are shown as

\[
\begin{align*}
a &= a_0 + a_{11} U_1 + a_{12} U_2 + \frac{1}{\sqrt{p}} (a_{21} U_1^2 + a_{22} U_1 U_2 + a_{23} U_2^2 + a_{24})
+ \frac{1}{p} (a_{31} U_1^3 + a_{32} U_1^2 U_2 + a_{33} U_1 U_2^2 + a_{34} U_2^3 + a_{35} U_1 + a_{36} U_2) + O(p^{-1}),
\end{align*}
\]

\[
\begin{align*}
\frac{\kappa_3}{6v_p^3} &= \frac{1}{\sqrt{p}} \beta_0 + \frac{1}{p} \left(\beta_{11} U_1 + \beta_{12} U_2\right) + O(p^{-1}),
\frac{\kappa_4}{24v_p^3} &= \frac{1}{p} \gamma_0 + O(p^{-1}),
\frac{\kappa_3^3}{72v_p^3} &= \frac{1}{p} \delta_0 + O(p^{-1}),
\end{align*}
\]

where

\[
\begin{align*}
a_0 &= -\frac{r_0}{\sqrt{s_0}}, \quad a_{11} = \frac{r_0 s_{11}}{2\sqrt{s_0^3}} - \frac{r_{11}}{\sqrt{s_0}}, \quad a_{12} = \frac{r_0 s_{12}}{2\sqrt{s_0^3}} - \frac{r_{12}}{\sqrt{s_0}},
\end{align*}
\]

\[
\begin{align*}
a_{21} &= -\frac{3r_0 s_{11}^2}{4\sqrt{s_0}} + \frac{2r_0 s_{21} + s_{11} r_{11}}{2\sqrt{s_0^3}} - \frac{r_{21}}{\sqrt{s_0}},
\end{align*}
\]

\[
\begin{align*}
a_{22} &= -\frac{3r_0 s_{11} s_{12}}{4\sqrt{s_0^5}} + \frac{r_0 s_{22} + s_{12} r_{11} + r_{12} s_{11}}{2\sqrt{s_0^3}},
\end{align*}
\]

\[
\begin{align*}
a_{23} &= -\frac{3r_0 s_{12}^2}{4\sqrt{s_0^3}} + \frac{2r_0 s_{23} + s_{12} r_{12}}{2\sqrt{s_0^3}} - \frac{r_{22}}{\sqrt{s_0}}, \quad a_{24} = \frac{r_0 s_{24}}{2\sqrt{s_0^3}} - \frac{r_{23} + 2\log c}{\sqrt{s_0}},
\end{align*}
\]

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\[
\begin{align*}
\alpha_{31} &= \frac{15r_0s_{11}^3}{8\sqrt{s_0}} - \frac{18r_0s_{11}s_{21} + 3r_{12}s_{11}^2}{4\sqrt{s_0}} + \frac{6r_0s_{31} + 2s_{21}r_{11} + r_{21}s_{11}}{2\sqrt{s_0}} - \frac{r_{31}}{\sqrt{s_0}}, \\
\alpha_{32} &= \frac{15r_0s_{12}^2 s_{11}}{8\sqrt{s_0}} - \frac{6r_0(s_{12}s_{21} + s_{11}s_{22}) + 3r_{12}s_{11}^2 + 3r_{11}s_{11}s_{12}}{4\sqrt{s_0}} \\
&\quad + \frac{2r_0s_{12} + 2s_{21}r_{11} + r_{11}s_{12} + r_{21}s_{12}}{2\sqrt{s_0}}, \\
\alpha_{33} &= \frac{15r_0s_{12}^2 s_{12}}{8\sqrt{s_0}} - \frac{6r_0(s_{12}s_{22} + s_{11}s_{23}) + 3r_{11}s_{12}^2 + 3r_{12}s_{11}s_{12}}{4\sqrt{s_0}} \\
&\quad + \frac{2r_0s_{12} + 2s_{23}r_{11} + r_{12}s_{22} + r_{22}s_{12}}{2\sqrt{s_0}}, \\
\alpha_{34} &= \frac{15r_0s_{12}^3}{8\sqrt{s_0}} - \frac{18r_0s_{12}s_{23} + 3r_{12}s_{12}^2}{4\sqrt{s_0}} + \frac{6r_0s_{34} + 2s_{23}r_{12} + r_{23}s_{12}}{2\sqrt{s_0}} - \frac{r_{34}}{\sqrt{s_0}}, \\
\alpha_{35} &= \frac{3r_0s_{11}s_{24}}{4\sqrt{s_0}} - \frac{r_0s_{35} + s_{24}r_{11} + s_{11}r_{23}}{2\sqrt{s_0}}, \\
\alpha_{36} &= \frac{3r_0s_{12}s_{24}}{4\sqrt{s_0}} - \frac{r_0s_{36} + s_{24}r_{12} + s_{12}r_{23}}{2\sqrt{s_0}}, \\
\beta_0 &= \frac{t_0}{6\sqrt{p}}, \quad \beta_{11} = \frac{s_{11}t_0}{4\sqrt{s_0}} + \frac{t_{11}}{6s_0}, \\
\beta_{12} &= \frac{s_{12}t_0}{4\sqrt{s_0}} + \frac{t_{12}}{6s_0}, \\
\gamma_0 &= \frac{u_0}{24s_0}, \quad \delta_0 = \frac{t_0^3}{72s_0^3}.
\end{align*}
\]

Therefore, we can calculate the right side of (2.17) as the expectation of \( U_i \).

\[
f_{U_i}(x) := \phi(x)\left\{1 + \frac{1}{\sqrt{p}}\tau_i(x^3 - 3x) + \frac{1}{p}\tau_{2i}(2x^6 - 21x^4 + 36x^2 - 3)\right\} + o(p^{-1}),
\]

(A.1)

where

\[
\tau_i = \frac{\sqrt{2\sqrt{p}}}{3\sqrt{n_i - p + 1}}, \quad \tau_{2i} = \frac{p}{18(n_i - p + 1)},
\]

and some property about standard normal distribution function. Then, we obtain the parameters \( f_1, f_2 \) and \( f_3 \) of Theorem 2.4,

\[
f_1 = \Phi \left( -\frac{a_0}{\sqrt{\alpha_{11}^2 + \alpha_{12}^2} + 1} \right),
\]

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\[ f_2 = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{\alpha_0^2}{\alpha_{11}^2 + \alpha_{12}^2 + 1}\right) \frac{1}{\sqrt{\alpha_{11}^2 + \alpha_{12}^2 + 1}} \]
\[
\times \left\{ \frac{\alpha_{11} \alpha_0}{\alpha_{11}^2 + \alpha_{12}^2 + 1} \frac{\alpha_1}{\alpha_{11}^2 + \alpha_{12}^2 + 1} - \frac{\alpha_{12} \alpha_0}{\alpha_{11}^2 + \alpha_{12}^2 + 1} \frac{\alpha_2}{\alpha_{11}^2 + \alpha_{12}^2 + 1} \right\} \\
+ \frac{(\alpha_{12}^2 + 1)^2 + \alpha_{11}^2 (1 + \alpha_0^2 + \alpha_{12}^2)}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^2} \theta_{12} \frac{\alpha_{11} \alpha_{12} (1 - \alpha_0^2 + \alpha_{11}^2 + \alpha_{12}^2)}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^2} \theta_{21} \\
+ \frac{(\alpha_{11}^2 + 1)^2 + \alpha_{12}^2 (1 + \alpha_0^2 + \alpha_{11}^2)}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^2} \theta_{23} \right\},
\]
\[ f_3 = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{\alpha_0^2}{\alpha_{11}^2 + \alpha_{12}^2 + 1}\right) \frac{1}{\sqrt{\alpha_{11}^2 + \alpha_{12}^2 + 1}} \]
\[
\times \left\{ \frac{\alpha_{11} \alpha_0}{\alpha_{11}^2 + \alpha_{12}^2 + 1} \frac{\alpha_1}{\alpha_{11}^2 + \alpha_{12}^2 + 1} - \frac{\alpha_{12} \alpha_0}{\alpha_{11}^2 + \alpha_{12}^2 + 1} \frac{\alpha_2}{\alpha_{11}^2 + \alpha_{12}^2 + 1} \right\} \\
- \frac{\alpha_{11} \alpha_{12} (1 - \alpha_0^2 + \alpha_{11}^2 + \alpha_{12}^2)}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^3} \theta_{21} \\
- \frac{(\alpha_{11}^2 + 1)^2 + \alpha_{12}^2 (1 + \alpha_0^2 + \alpha_{11}^2)}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^2} \theta_{22} \\
+ \left[ 3 (1 + \alpha_{12}^2)^2 + \alpha_{11}^2 (1 + \alpha_0^2 - \alpha_{12}^2) - 2 \alpha_0^4 \right] \theta_{31} \\
+ \frac{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^3}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^4} \theta_{32} \\
- \frac{(1 + \alpha_{11}^2)^2 + \alpha_{12}^2 (1 + \alpha_0^2 - \alpha_{11}^2)}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^3} \theta_{33} \\
- \frac{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^3}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^4} \theta_{34} \\
+ \left[ \alpha_{11} \alpha_{12} \left\{ 3 (1 + \alpha_{12}^2)^2 + 6 \alpha_{11} (1 + \alpha_{12}^2)^2 (1 + \alpha_0^2) + \alpha_{11}^2 \right\} \right] \theta_{41} \\
+ \frac{\alpha_{11} \alpha_{12} \left\{ 3 (1 + \alpha_{12}^2)^2 + 6 \alpha_{11} (1 + \alpha_{12}^2)^2 (1 + \alpha_0^2) + 3 (1 + \alpha_{12}^2)^2 \right\}}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^4} \theta_{42} \\
+ \frac{\alpha_{11} \alpha_{12} \left\{ \alpha_{11}^2 + \alpha_{12}^2 + 1 \right\}}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^4} \theta_{43} \\
+ \frac{\alpha_{11} \alpha_{12} \left\{ \alpha_{11}^2 + \alpha_{12}^2 + 1 \right\}}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^4} \theta_{44} \right\} \right\},
\]
\[ 21 \]
\[
\begin{align*}
\theta_{43} & = \frac{\alpha_1^4 + 9\alpha_3^4 + 6\alpha_2^4 + \alpha_0^2 (2 - 3\alpha_2^4)}{(\alpha_1^2 + \alpha_2^2 + 1)^4} \\
\theta_{44} & = \frac{\alpha_1^6 + \alpha_2^4 (\alpha_0^2 + 3(1 + \alpha_2^2))(1 + \alpha_2^2 + \alpha_0^2)^2 - 3\alpha_0^2 (\alpha_1^4 + \alpha_2^4 - (1 + \alpha_2^2))}{(\alpha_1^2 + \alpha_2^2 + 1)^4} \\
\theta_{45} & = \frac{\alpha_1^4 + 6\alpha_2^2 (1 + \alpha_0^2)(2 - 3\alpha_2^4)}{(\alpha_1^2 + \alpha_2^2 + 1)^4} \\
\theta_{51} & = \frac{\alpha_1^4 + \alpha_2^4 (1 + \alpha_0^2) + 3(1 + \alpha_2^2)}{(\alpha_1^2 + \alpha_2^2 + 1)^4} \\
\theta_{52} & = \frac{\alpha_1^4 + 2\alpha^2 (1 + \alpha_2^2) - 21(1 + \alpha_2^2)}{(\alpha_1^2 + \alpha_2^2 + 1)^5} \\
\theta_{53} & = \frac{\alpha_1^4 (3 + \alpha_2^4 + 9\alpha_2^2) - 3(1 + \alpha_2^2)(1 - (1 + \alpha_2^2)(\alpha_1^2 + 2\alpha_2^4))}{(\alpha_1^2 + \alpha_2^2 + 1)^5} \\
\theta_{54} & = \frac{\alpha_1^4 + \alpha_2^4 (2 - 5\alpha_2^2) + 3(3 + 7\alpha_2^2 + 4\alpha_4^4)}{(\alpha_1^2 + \alpha_2^2 + 1)^5} \\
\end{align*}
\]
\[
\begin{align*}
&+3(1 + \alpha_{11}^2)^4 - 4\alpha_{12}^2(3 + \alpha_{11}^2 + 3\alpha_{11}^2) \\
&\quad \left(\frac{\alpha_{11}^2 + \alpha_{12}^2 + 1}{1 + \alpha_{11}^2\alpha_{12}^2 + \alpha_{12}^2}ight) \theta_{55} \\
&+ \frac{\alpha_0\alpha_{11}\{15(1 + \alpha_{11}^2)^4 + 10\alpha_{12}^2(1 + \alpha_{11}^2)^2(3 + \alpha^2 + 3\alpha_{11}^2)\}}{(\alpha_{11}^2 + \alpha_{12}^2 + 1)^5} \theta_{56}
\end{align*}
\]

where
\[
\begin{align*}
\zeta_0 &= -\alpha_{24} + \beta_0 - \alpha_0^3\beta_0 + 3\alpha_{11}\tau_{11} + \alpha_{12}\tau_{12}, \quad \zeta_{11} = -2\alpha_0\beta_0\alpha_{11}, \\
\zeta_{12} &= -2\alpha_0\beta_0\alpha_{12}, \quad \zeta_{21} = -\alpha_{21} - \alpha_{11}^2\beta_0 - \alpha_{11}\tau_{11}, \\
\zeta_{22} &= -\alpha_{22} - 2\alpha_{11}\alpha_{12}\beta_0, \quad \zeta_{23} = -\alpha_{23} - \alpha_{12}^2\beta_0 - \alpha_{12}\tau_{12}, \\
\theta_0 &= \frac{\alpha_0\alpha_{24}^2}{2} - 3\alpha_0\alpha_{24}\beta_0 + 3\alpha_0^3\alpha_{24}\beta_0 \\
&- 15\alpha_0\delta_0 + 10\alpha_0^3\delta_0 - \alpha_0^5\delta_0 + 3\alpha_0^3\gamma_0 - \alpha_0^3\gamma_0, \\
\theta_{11} &= \frac{\alpha_{11}\alpha_{24}^2}{2} - \alpha_{35} - 3\alpha_{11}\alpha_{24}\beta_0 + 3\alpha_0\alpha_{11}\alpha_{12}\beta_0 + \beta_{11} \\
&- \alpha_0^2\beta_{11} - 15\alpha_{11}\delta_0 + 30\alpha_0^2\alpha_{11}\delta_0 - 5\alpha_0^4\alpha_{11}\delta_0 + 3\alpha_{11}\gamma_0 - 3\alpha_0^3\alpha_{11}\gamma_0 \\
&+ 3\alpha_{24}\tau_{11} - 3\beta_0\tau_{11} + 3\alpha_0^2\beta_0\tau_{11} - 3\alpha_{11}\tau_{21}, \\
\theta_{12} &= \frac{\alpha_{12}\alpha_{24}^2}{2} - \alpha_{36} - 3\alpha_{12}\alpha_{24}\beta_0 + 3\alpha_0\alpha_{12}\alpha_{24}\beta_0 + \beta_{12} \\
&- \alpha_0^2\beta_{12} - 15\alpha_{12}\delta_0 + 30\alpha_0^2\alpha_{12}\delta_0 - 5\alpha_0^4\alpha_{12}\delta_0 + 3\alpha_{12}\gamma_0 - 3\alpha_0^3\alpha_{12}\gamma_0 \\
&+ 3\alpha_{24}\tau_{12} - 3\beta_0\tau_{12} + 3\alpha_0^2\beta_0\tau_{12} - 3\alpha_{12}\tau_{22}, \\
\theta_{21} &= \alpha_0\alpha_{24}\beta_0 - 3\alpha_0\alpha_{24}\beta_0 + \alpha_0^2\alpha_{24}\beta_0 + 3\alpha_0\alpha_{11}^2\alpha_{24}\beta_0 \\
&- 2\alpha_0\alpha_{11}\beta_{11} + 3\alpha_0\alpha_{11}\delta_0 - 10\alpha_0^3\alpha_{11}\delta_0 - 3\alpha_0\alpha_{11}\gamma_0 + 6\alpha_0\alpha_{11}\beta_{11}\tau_{11}, \\
\theta_{22} &= \alpha_0\alpha_{24}\beta_0 - 3\alpha_0\alpha_{24}\beta_0 + \alpha_0^2\alpha_{24}\beta_0 + 6\alpha_0\alpha_{11}\alpha_{12}\alpha_{24}\beta_0 \\
&- 2\alpha_0\alpha_{12}\beta_{11} - 2\alpha_0\alpha_{11}\beta_{12} + 6\alpha_0\alpha_{11}\alpha_{12}\delta_0 - 20\alpha_0^3\alpha_{11}\alpha_{12}\delta_0 \\
&- 6\alpha_0\alpha_{11}\alpha_{12}\gamma_0 + 6\alpha_0\alpha_{12}\beta_{11}\tau_{11} + 6\alpha_0\alpha_{11}\beta_{12}\tau_{12}, \\
\theta_{23} &= \alpha_0\alpha_{24}\beta_0 - 3\alpha_0\alpha_{24}\beta_0 + \alpha_0^2\alpha_{24}\beta_0 + 3\alpha_0\alpha_{12}\beta_{12}\beta_0 \\
&- 2\alpha_0\alpha_{11}\beta_{12} + 3\alpha_0\alpha_{12}\beta_{12}\delta_0 - 10\alpha_0^3\alpha_{12}\delta_0 - 3\alpha_0\alpha_{12}\gamma_0 + 6\alpha_0\alpha_{12}\beta_{12}\tau_{12}, \\
&+ \text{lower terms},
\end{align*}
\]
\[ \theta_{31} = \alpha_{11} \alpha_{21} \alpha_{24} - \alpha_{31} - 3 \alpha_{11} \alpha_{21} \beta_0 + 3 \alpha_0^2 \alpha_{11} \alpha_{21} \beta_0 + \alpha_1^3 \alpha_{24} \beta_0 \]
\[-\alpha_{11}^2 \beta_1 + 10 \alpha_1^3 \delta_0 - 10 \alpha_0^2 \alpha_{11}^3 \delta_0 - \alpha_3^2 \gamma_0 \]
\[+3 \alpha_{21} \tau_{11} - \alpha_{21} \tau_{11} + \beta_0 \tau_{11} - \alpha_0^2 \beta_0 \tau_{11} + 3 \alpha_0^2 \beta_0 \tau_{11} + 11 \alpha_{11} \tau_{21} \]
\[\theta_{32} = \alpha_{12} \alpha_{21} \alpha_{24} + \alpha_{11} \alpha_{22} \alpha_{24} - \alpha_{32} \]
\[-3 \alpha_{12} \alpha_{21} \beta_0 + 3 \alpha_0^2 \alpha_{12} \alpha_{21} \beta_0 - 3 \alpha_{11} \alpha_{22} \beta_0 + 3 \alpha_0^2 \alpha_{11} \alpha_{22} \beta_0 + 3 \alpha_1^2 \alpha_{12} \alpha_{24} \beta_0 \]
\[-2 \alpha_{11} \alpha_{12} \beta_1 + \alpha_1^2 \beta_1 + 3 \alpha_1^2 \alpha_{12} \delta_0 - 3 \alpha_0^2 \alpha_{11} \alpha_{12} \delta_0 - \alpha_1^2 \alpha_{12} \gamma_0 \]
\[+3 \alpha_{22} \tau_{11} + 6 \alpha_{11} \alpha_{12} \beta_0 \tau_{11} + 3 \alpha_{21} \tau_{12} + 3 \alpha_{11} \beta_0 \tau_{12} \]
\[\theta_{33} = \alpha_{12} \alpha_{22} \alpha_{24} + \alpha_{11} \alpha_{22} \alpha_{24} - \alpha_{33} \]
\[-3 \alpha_{12} \alpha_{22} \beta_0 + 3 \alpha_0^2 \alpha_{12} \alpha_{22} \beta_0 - 3 \alpha_{11} \alpha_{23} \beta_0 + 3 \alpha_0^2 \alpha_{11} \alpha_{23} \beta_0 + 3 \alpha_1^2 \alpha_{12} \alpha_{24} \beta_0 \]
\[-2 \alpha_{11} \alpha_{12} \beta_1 + \alpha_1^2 \beta_1 + 3 \alpha_1^2 \alpha_{12} \delta_0 - 3 \alpha_0^2 \alpha_{11} \alpha_{12} \delta_0 - \alpha_1^2 \alpha_{12} \gamma_0 \]
\[+3 \alpha_{23} \tau_{11} + 6 \alpha_{11} \alpha_{12} \beta_0 \tau_{11} + 3 \alpha_{22} \tau_{12} + 3 \alpha_{12} \beta_0 \tau_{11} \]
\[\theta_{34} = \alpha_{12} \alpha_{23} \alpha_{24} - \alpha_{34} - 3 \alpha_{12} \alpha_{23} \beta_0 + 3 \alpha_0^2 \alpha_{12} \alpha_{23} \beta_0 + \alpha_1^2 \alpha_{24} \beta_0 \]
\[-\alpha_1^2 \beta_1 + 10 \alpha_1^3 \delta_0 - 10 \alpha_0^2 \alpha_{12}^3 \delta_0 - \alpha_3^2 \gamma_0 \]
\[+3 \alpha_{23} \tau_{12} - \alpha_{23} \tau_{12} + \beta_0 \tau_{12} - \alpha_0^2 \beta_0 \tau_{12} + 3 \alpha_0^2 \beta_0 \tau_{12} + 11 \alpha_{12} \tau_{22} \]
\[\theta_{41} = \frac{\alpha_0^2 \alpha_1^2}{2} + 3 \alpha_0 \alpha_{11} \alpha_{21} \beta_0 - 5 \alpha_0 \alpha_{11}^4 \beta_0 - 2 \alpha_0 \alpha_{11} \beta_0 \tau_{11} \]
\[\theta_{42} = \alpha_0 \alpha_{21} \alpha_{23} + 6 \alpha_0 \alpha_{11} \alpha_{12} \alpha_{21} \beta_0 + 3 \alpha_0 \alpha_{11} \alpha_{21} \beta_0 \]
\[\theta_{43} = \frac{\alpha_0 \alpha_1^2}{2} + \alpha_0 \alpha_{21} \alpha_{23} + 3 \alpha_0 \alpha_{12} \alpha_{21} \beta_0 \]
\[+6 \alpha_0 \alpha_{11} \alpha_{12} \alpha_{22} \beta_0 + 3 \alpha_0 \alpha_{11} \alpha_{23} \beta_0 - 3 \alpha_0 \alpha_{11} \alpha_{12} \beta_0 \tau_{12} \]
\[\theta_{44} = \alpha_0 \alpha_{22} \alpha_{23} + 6 \alpha_0 \alpha_{11} \alpha_{12} \alpha_{23} \beta_0 + 3 \alpha_0 \alpha_{12} \alpha_{22} \beta_0 \]
\[\theta_{45} = \frac{\alpha_0 \alpha_1^2}{2} + \alpha_0 \alpha_{11} \alpha_{22} \beta_0 - 5 \alpha_0 \alpha_1^4 \beta_0 - 2 \alpha_0 \alpha_{11} \beta_0 \tau_{12} \]
\[\theta_{51} = \frac{\alpha_1 \alpha_2}{2} + \alpha_1 \alpha_{21} \beta_0 - \alpha_1 \beta_0 \tau_{11} - \alpha_1 \beta_0 \tau_{11} - 2 \alpha_1 \tau_{21} \]
\[\theta_{52} = \frac{\alpha_1 \alpha_2}{2} + \alpha_1 \alpha_{21} \beta_2 + 3 \alpha_1 \alpha_{12} \alpha_{21} \beta_0 + \alpha_1 \alpha_{22} \beta_0 \]

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\[-5\alpha_1^4 \alpha_{12}\delta_0 - \alpha_{22}\tau_{11} - 2\alpha_{11}\alpha_{12}/\beta_0\tau_{11},\]
\[\theta_{33} = \frac{\alpha_{11}\alpha_{22}}{2} + \alpha_{12}\alpha_{21}\alpha_{22} + \alpha_{11}\alpha_{21}\alpha_{23} + 3\alpha_{11}\alpha_{12}\alpha_{21}\beta_0\]
\[+3\alpha_{11}^2\alpha_{12}\alpha_{22}/\beta_0 + \alpha_{12}^3/\alpha_{21}\beta_0 - 10\alpha_{11}^3/\alpha_{12}\delta_0 - \alpha_{23}\tau_{11} - \alpha_{12}^2/\beta_0\tau_{11},\]
\[\theta_{34} = \frac{\alpha_{12}\alpha_{22}}{2} + \alpha_{12}\alpha_{21}\alpha_{23} + \alpha_{11}\alpha_{22}\alpha_{23} + 3\alpha_{11}\alpha_{12}\alpha_{22}\beta_0\]
\[+3\alpha_{11}^2\alpha_{12}\alpha_{23}/\beta_0 + \alpha_{12}^3/\alpha_{21}\beta_0 - 10\alpha_{11}^3/\alpha_{12}\delta_0 - \alpha_{23}\tau_{12} - \alpha_{11}\alpha_{12}/\beta_0\tau_{12},\]
\[\theta_{35} = \frac{\alpha_{11}\alpha_{22}}{2} + \alpha_{12}\alpha_{22}\alpha_{23} + 3\alpha_{11}\alpha_{12}\alpha_{23}/\beta_0 + \alpha_{12}^3/\alpha_{22}\beta_0\]
\[-5\alpha_{11}^2/\alpha_{12}\delta_0 - 2\alpha_{11}\alpha_{12}/\beta_0\tau_{12},\]
\[\theta_{36} = \frac{\alpha_{12}\alpha_{22}}{2} + \alpha_{12}^3/\alpha_{22}\beta_0 - \alpha_{23}\tau_{12} - \alpha_{12}/\beta_0\tau_{12} - 2\alpha_{12}/\tau_{22}.\]

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**Reference**


*Chieko Matsumoto*

Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo, 186-8603, Japan

*e-mail* shisato@ier.hit-u.ac.jp

*Hirofumi Wakaki*

Department of Mathematics
Graduate school of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan

*e-mail* wakaki@math.sci.hiroshima-u.ac.jp