A Borsuk-Ulam type theorem and its application to combinatorics

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Let G be a simple graph and p, q integers satisfying $1 \leqq q \leqq p$. We denote by $V(G)$ the vertices set of G. A *(p, q)-coloring* of G is a map c: $V(G) \to [p]$ such that $q \leqq |c(x) - c(y)| \leqq p - q$ for every edge xy of G. The *circular chromatic number* of G is

$$\chi_c(G) = \inf\{p/q \mid \text{There exists a } (p, q)\text{-coloring of } G\}.$$

As the ordinary chromatic number $\chi(G)$ is equal to

$$\min\{p \mid \text{There exists a } (p, 1)\text{-coloring of } G\},$$

we easily see $\chi_c(G) \leqq \chi(G)$. It has been known $\chi_c(G) > \chi(G) - 1$.

Let n and k be integers satisfying $n \geqq 2k$. We denote by $\binom{[n]}{k}$ the collection of all subsets of $[n]$ with cardinary k. The *Kneser graph* $KG_{n,k}$ for $n \geqq 2k > 0$, has vertex set $\binom{[n]}{k}$ and any two vertices $u, v \in \binom{[n]}{k}$ are adjacent if and only if $u \cap v = \emptyset$.

In [4], Lovász proved $\chi(KG_{n,k}) = n - 2k + 2$ by using Borsuk-Ulam theorem. Chen proved the following theorem in [3].

Theorem 1 ([3]) $\chi_c(KG_{n,k}) = \chi(KG_{n,k})(= n - 2k + 2)$

In this talk, we give a topological proof of this theorem.

Next we consider the chromatic number of multiple Kneser hypergraphs. Let r, k, m, n be positive integers where $r \geqq 2$. Let $\pi = (P_1, \cdots, P_m)$ be a partition of $[n]$ and $\vec{s} = (s_1, \cdots, s_m)$ a positive integer vector. The *multiple Kneser hypergraph* $KG^r(\pi; \vec{s}; k)$ is a r-unirorm hypergraph with the vertex set

$$V := \{A \subset [n] \mid |A| = k,\ 1 \leqq \forall i \leqq m;\ |A \cap P_i| \leqq s_i\},$$

where $\{A_1, \dots A_r\}$ is a hyperedge if $A_1, \dots, A_r$ are pairwise disjoint. We define the function $f_{2,\pi}$ as follows

$$f_{2,\pi}(P_i) := \min\{2s_i,\ |P_i|\}.$$

We set

$$M_{2,\pi} = \max\Big\{2k - 1 + \sum_{j=1}^{t}(|P_{i_j}| - f_{2,\pi}(P_{i_j})) \mid \sum_{j=1}^{t} f_{2,\pi}(P_{i_j}) \leq 2k - 1\Big\},$$

We give a topological proof of the following theorem which has been proved in [1].

Theorem 2 ([1]) $\chi(KG^2(\pi; \vec{s}; k)) = n - M_{2,\pi} + 1$

References

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