A model for the $K(n)$-local stable homotopy category

Takeshi Torii
(Okayama University)

The $K(n)$-local stable homotopy category occupies an important position in chromatic homotopy theory that is a branch of stable homotopy theory, where $K(n)$ is the Morava K-theory. For a spectrum X, we denote by $L_{K(n)}X$ the $K(n)$-localization of X. For a finite spectrum Y, we have the $K(n)$-local E_n-based Adams spectral sequence

$$E_2^{s,t} = H_c^s(\mathbb{G}_n; (E_n)_t(Y)) \Longrightarrow \pi_{t-s}(L_{K(n)}Y),$$

where E_n is the Morava E-theory and $\mathbb{G}_n$ is the Morava stabilizer group. This suggests that the $K(n)$-local stable homotopy category be related to the derived category of $\mathbb{G}_n$-equivariant E_n-modules. In this talk I am going to construct a model for the $K(n)$-local stable homotopy category by means of the Morava E-theory and the Morava stabilizer group.

Since the stabilizer group $\mathbb{G}_n$ is a profinite group, we can consider the category $\Sigma\mathrm{Sp}(\mathbb{G}_n)$ of discrete symmetric $\mathbb{G}_n$-spectra which are constructed based on simplicial discrete $\mathbb{G}_n$-sets. The category $\Sigma\mathrm{Sp}(\mathbb{G}_n)$ supports a model structure and we denote by $\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}$ its left Bousfield localization with respect to $K(n)$. We can construct a commutative monoid object $F_n \in \Sigma\mathrm{Sp}(\mathbb{G}_n)$ which satisfies $L_{K(n)}F_n \simeq E_n$.

Theorem 1 ([1])**.** The left Quillen functor

$$F_n \wedge (-) : \Sigma\mathrm{Sp}_{K(n)} \to \mathrm{Mod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$$

is a Quillen equivalence. In particular, we see that the $K(n)$-local stable homotopy category $\mathrm{Ho}(\Sigma\mathrm{Sp}_{K(n)})$ is equivalent to the homotopy category of the model category $\mathrm{Mod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)})$ (as tensor triangulated categories).

$$\mathrm{Ho}(\Sigma\mathrm{Sp}_{K(n)}) \simeq \mathrm{Ho}(\mathrm{Mod}_{F_n}(\Sigma\mathrm{Sp}(\mathbb{G}_n)_{K(n)}))$$

References

[1] T. Torii, On quasi-categories of comodules and Landweber exactness, preprint, arXiv:1612.03265.