Affine definable $C^\infty G$ manifold structures in an o-minimal structure

Tomohiro Kawakami

Wakayama University

November 17, 2017

Ordered fields

- A field $(\boldsymbol{R}, +, \cdot, <)$ with a dense linear order $<$ without endpoints is an ordered field if it satisfies the following two conditions.

Ordered fields

- A field $(\boldsymbol{R}, +, \cdot, <)$ with a dense linear order $<$ without endpoints is an ordered field if it satisfies the following two conditions.
  (1) For any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{R}$, if $\boldsymbol{x} < \boldsymbol{y}$, then $\boldsymbol{x} + \boldsymbol{z} < \boldsymbol{y} + \boldsymbol{z}$.
  (2) For any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{R}$, if $\boldsymbol{x} < \boldsymbol{y}$ and $\boldsymbol{z} > \boldsymbol{0}$, then $\boldsymbol{xz} < \boldsymbol{yz}$.

Real closed fields

- An ordered field $(R, +, \cdot, <)$ is a *real field* if it satisfies the following condition.

- An ordered field $(R, +, \cdot, <)$ is a *real field* if it satisfies the following condition.
  For any $y_1, \dots, y_m \in R$,
  $y_1^2 + \cdots + y_m^2 = 0 \Rightarrow y_1 = \cdots = y_m = 0$.

Real closed fields

- An ordered field $(R, +, \cdot, <)$ is a *real field* if it satisfies the following condition.
  For any $y_1, \dots, y_m \in R$,
  $y_1^2 + \cdots + y_m^2 = 0 \Rightarrow y_1 = \cdots = y_m = 0$.
  A real field $(R, +, \cdot, <)$ is a *real closed field* if it satisfies one of the following two equivalent conditions.

- An ordered field $(R, +, \cdot, <)$ is a *real field* if it satisfies the following condition.
  For any $y_1, \dots, y_m \in R$,
  $y_1^2 + \cdots + y_m^2 = 0 \Rightarrow y_1 = \cdots = y_m = 0$.
  A real field $(R, +, \cdot, <)$ is a *real closed field* if it satisfies one of the following two equivalent conditions.
  (1) [Intermediate value property] For every $f(x) \in R[x]$, if $a < b$ and $f(a) \neq f(b)$, then $f([a, b]_R)$ contains $[f(a), f(b)]_R$ if $f(a) < f(b)$ or $[f(b), f(a)]_R$ if $f(b) < f(a)$, where $[a, b]_R = \{x \in R | a \leq x \leq b\}$.
  (2) The ring $R[i] = R[x]/(x^2 + 1)$ is an algebraically closed field.

O-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbf{R} = (\mathbb{R}, +, \cdot, <)$ of real numbers.

O-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
(1) The field $\mathbf{R} = (\mathbb{R}, +, \cdot, <)$ of real numbers.
(2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where f ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.

O-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbf{R} = (\mathbb{R}, +, \cdot, <)$ of real numbers.
  (2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where f ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.
  (3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where S is a subset of $\mathbb{R}$, f ranges over all restricted analytic functions as in (2), and the function $x^r : \mathbb{R} \to \mathbb{R}$ is given by

$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \leq 0 \end{cases}.$$

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbf{R} = (\mathbb{R}, +, \cdot, <)$ of real numbers.
  (2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where f ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.
  (3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where S is a subset of $\mathbb{R}$, f ranges over all restricted analytic functions as in (2), and the function $x^r : \mathbb{R} \to \mathbb{R}$ is given by
$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \leq 0 \end{cases}.$$
  (4) $\mathbf{R}_{exp} := (\mathbb{R}, +, \cdot, <, exp)$, where $exp : \mathbb{R} \to \mathbb{R}$ denotes the exponential function $x \mapsto e^x$.

O-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
(1) The field $\mathbf{R} = (\mathbb{R}, +, \cdot, <)$ of real numbers.
(2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where f ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.
(3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where S is a subset of $\mathbb{R}$, f ranges over all restricted analytic functions as in (2), and the function $x^r : \mathbb{R} \to \mathbb{R}$ is given by

$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \leq 0 \end{cases} .$$

(4) $\mathbf{R}_{exp} := (\mathbb{R}, +, \cdot, <, exp)$, where $exp : \mathbb{R} \to \mathbb{R}$ denotes the exponential function $x \mapsto e^x$.
(5) $\mathbf{R}_{an,exp} := (\mathbb{R}, +, \cdot, <, (f), exp)$, where (f) and exp denote as above.

O-minimal structures

- An ordered structure $(R, <)$ with a dense linear order $<$ without endpoints is *o-minimal (order minimal)* if every definable set of R is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.

- An ordered structure $(R, <)$ with a dense linear order $<$ without endpoints is *o-minimal (order minimal)* if every definable set of R is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.
  If $(R, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.

O-minimal structures

- An ordered structure $(\boldsymbol{R}, <)$ with a dense linear order $<$ without endpoints is *o-minimal (order minimal)* if every definable set of $\boldsymbol{R}$ is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.
  If $(\boldsymbol{R}, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.
  The topology of $\boldsymbol{R}$ is the interval topology and the topology of $\boldsymbol{R}^n$ is the product topology.

- An ordered structure $(\boldsymbol{R}, <)$ with a dense linear order $<$ without endpoints is *o-minimal (order minimal)* if every definable set of $\boldsymbol{R}$ is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.
  If $(\boldsymbol{R}, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.
  The topology of $\boldsymbol{R}$ is the interval topology and the topology of $\boldsymbol{R}^n$ is the product topology.
  In this presentation, everything is considered in an exponential o-minimal expansion $\mathcal{N} = (\mathbb{R}, +, \cdot, <, e^x, \dots,)$ with the C^∞ cell decomposition of the field of real numbers $(\mathbb{R}, +, \cdot, <)$ unless otherwise stated.

Definable G sets and definable G maps

- Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f $(\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m)$ is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

Definable G sets and definable G maps

- Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f $(\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m)$ is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.
  Let G be a compact subgroup of $GL_n(\mathbb{R})$.

Definable G sets and definable G maps

- Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f $(\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m)$ is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.
  Let G be a compact subgroup of $GL_n(\mathbb{R})$.
  A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a *definable G set* if ϕ is definable. We simply write X instead of (X, ϕ) and *gx* instead of $\phi(g, x)$.

Definable G sets and definable G maps

- Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m$) is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.
  Let G be a compact subgroup of $GL_n(\mathbb{R})$.
  A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a *definable G set* if ϕ is definable. We simply write X instead of (X, ϕ) and *gx* instead of $\phi(g, x)$.
  A definable map $f : X \to Y$ between definable G sets is a *definable G map* if for any $x \in X, g \in G$, $f(gx) = gf(x)$. A definable G map is a *definable G homeomorphism* if it is a homeomorphism.

•

(1) A group homomorphism (resp. A group isomorphism) from G to $O_n(\mathbb{R})$ is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map (resp. a definable homeomorphism).
Note that a definable group homomorphism (resp. a definable group isomorphism) between G and $O_n(\mathbb{R})$ is a definable C^∞ map (resp. a definable C^∞ diffeomorphism) because G and $O_n(\mathbb{R})$ are Lie groups.

-

(1) A group homomorphism (resp. A group isomorphism) from G to $O_n(\mathbb{R})$ is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map (resp. a definable homeomorphism).
Note that a definable group homomorphism (resp. a definable group isomorphism) between G and $O_n(\mathbb{R})$ is a definable C^∞ map (resp. a definable C^∞ diffeomorphism) because G and $O_n(\mathbb{R})$ are Lie groups.

(2) An *n-dimensional representation* of G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. In this paper, we assume that every representation of G is orthogonal.

(3) A *definable C^rG manifold* is a pair (X, α) consisting of a definable C^r manifold X and a group action α of G on X such that $\alpha : G \times X \to X$ is a definable C^r map. For simplicity of notation, we write X instead of (X, α).

Definable C^rG manifolds

-

(1) A group homomorphism (resp. A group isomorphism) from G to $O_n(\mathbb{R})$ is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map (resp. a definable homeomorphism).
Note that a definable group homomorphism (resp. a definable group isomorphism) between G and $O_n(\mathbb{R})$ is a definable C^∞ map (resp. a definable C^∞ diffeomorphism) because G and $O_n(\mathbb{R})$ are Lie groups.

(2) An *n-dimensional representation* of G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. In this paper, we assume that every representation of G is orthogonal.

(3) A *definable C^rG manifold* is a pair (X, α) consisting of a definable C^r manifold X and a group action α of G on X such that $\alpha : G \times X \to X$ is a definable C^r map. For simplicity of notation, we write X instead of (X, α).

Definable C^rG manifolds

•

(4) A definable C^r submanifold of a definable C^rG manifold X is called a *definable C^rG submanifold* of X if it is G invariant.

(5) A definable C^r map (resp. A definable C^r diffeomorphism, A definable homeomorphism, A definable map) is a *definable C^rG map* (resp. a *definable C^rG diffeomorphism*, a *definable G homeomorphism*, a *definable G map*) if it is a G map.

(6) A definable C^rG manifold is called *affine* if it is definably C^rG diffeomorphic (definably G homeomorphic if $r = 0$) to a definable C^rG submanifold of some representation of G.

(7) A *definable C^rG manifold with boundary* is defined similarly.

•

(4) A definable C^r submanifold of a definable C^rG manifold X is called a *definable C^rG submanifold* of X if it is G invariant.

(5) A definable C^r map (resp. A definable C^r diffeomorphism, A definable homeomorphism, A definable map) is a *definable C^rG map* (resp. a *definable C^rG diffeomorphism*, a *definable G homeomorphism*, a *definable G map*) if it is a G map.

(6) A definable C^rG manifold is called *affine* if it is definably C^rG diffeomorphic (definably G homeomorphic if $r = 0$) to a definable C^rG submanifold of some representation of G.

(7) A *definable C^rG manifold with boundary* is defined similarly.

-

(4) A definable C^r submanifold of a definable C^rG manifold X is called a *definable C^rG submanifold* of X if it is G invariant.

(5) A definable C^r map (resp. A definable C^r diffeomorphism, A definable homeomorphism, A definable map) is a *definable C^rG map* (resp. a *definable C^rG diffeomorphism*, a *definable G homeomorphism*, a *definable G map*) if it is a G map.

(6) A definable C^rG manifold is called *affine* if it is definably C^rG diffeomorphic (definably G homeomorphic if $r = 0$) to a definable C^rG submanifold of some representation of G.

(7) A *definable C^rG manifold with boundary* is defined similarly.

Facts

Theorem

(1) If $\mathbf{0} \leq r < \infty$, then every definable C^r manifold is affine (2005).
(2) If $\mathcal{M}$ is exponential and G is a compact affine definable C^∞ group, then each compact definable $C^\infty G$ manifold is affine (1999).

-

- (1) A singleton $\{a\}$ is a *0-C^r cell* and an open interval (a, b) is a *1-C^r cell*.
  (2) Let C be a $(i_1, \dots i_n)$ cell and $f, h : C \to \mathbb{R} \cup \{\pm\infty\}$ with $f(x) < h(x)$ for all $x \in C$. The graph of f is an $(i_1, \dots i_n, 0)$ cell. The band $\{(x, y) \in C \times \mathbb{R} | f(x) < y < h(x)\}$ is an $(i_1, \dots i_n, 1)$ cell.
  (3) A partition $\{(-\infty, a_0), (a_0, a_1), \dots, (a_n, \infty), \{a_1\}, \dots, \{a_n\}\}$ is a *C^r cell decomposition* of $\mathbb{R}$.
  (4) A Partition $\mathcal{D}$ of $\mathbb{R}^n$ is *a C^r cell decomposition* of $\mathbb{R}^n$ if $\pi(\mathcal{D})$ is a C^r cell decomposition of $\mathbb{R}^{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection forgetting the last coordinate.
  (5) Let $A_1, \dots, A_k$ be definable subsets of $\mathbb{R}^n$. A *a C^r cell decomposition* $\mathcal{D}$ of $A_1, \dots, A_k$ if for each $C \in \mathcal{D}$, $C \subset A_i$ or $C \cap A_i = \emptyset$.

C^r Cell decompositions and exponential

Theorem (van den Dries 1998)

Let $A_1, \dots, A_k$ be definable subsets of $\mathbb{R}^n$. Then there exists a C^r cell decomposition $\mathcal{D}$ of $A_1, \dots, A_k$.

- We say that $\mathcal{M}$ admits a C^∞ *cell decomposition* if the above theorem is true when $r = \infty$.
  Remark that there exists an o-minimal structure which does not admit a C^∞ cell decomposition.
  We say that $\mathcal{M}$ is *polynomially bounded* if any definable function $f : \mathbb{R} \to \mathbb{R}$, there exist a positive integer N and $x_0 \in \mathbb{R}$ such that $|f(x)| < x^N$ for any $x > x_0$. If $e^x : \mathbb{R} \to \mathbb{R}$ is definable in $\mathcal{M}$, then $\mathcal{M}$ is *exponential*.

Theorem (Miller 1994)

Every o-minimal expansion of the field of real numbers is either polynomially bounded or exponential.

Approximation theorem

Theorem (2017)

If $0 \leq s < \infty$ and $\mathcal{M}$ admits C^∞ cell decomposition and exponential, then every definable $C^s G$ map between affine definable $C^\infty G$ manifolds is approximated in the definable C^s topology by definable $C^\infty G$ maps.

-

Theorem

Let X be an affine definable $C^r G$ manifold and $\mathcal{M}$ admits C^∞ cell decomposition and exponential. If $1 \leqq r < \infty$ then, X admits a unique affine definable $C^\infty G$ manifold structure up to definable $C^\infty G$ diffeomorphism.

-

Thank you very much.