Whitney approximation on smooth cell complexes

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Smooth relative CW complex

A pair (X, A) in $\mathbf{Diff}$ is called a smooth relative CW complex if there is a sequence of inclusions

$$A = X^{-1} \to X^0 \to \cdots \to X^{n-1} \to X^n \to \cdots$$

such that $A \to X$ coincides with $X^{-1} \to \operatorname{colim} X^n$, and for each $n \geq 0$ there are smooth maps $(\Phi_\lambda, \phi_\lambda) \colon (I^n, \partial I^n) \to (X^n, X^{n-1})$ $(\lambda \in \Lambda_n)$ which gives a diffeomorphism

$$X^n \cong \bigcup_{\lambda \in \Lambda_n} X^{n-1} \cup_{\phi_\lambda} I^n.$$

We call ϕ_λ and Φ_λ as attaching and characteristic maps.

X is simply called a smooth CW complex if $A = \emptyset$.

Main results

Proposition If (X, A) is a smooth relative CW complex then its image (TX, TA) under $T\colon \mathbf{Diff} \to \mathbf{Top}$ is a relative CW complex.

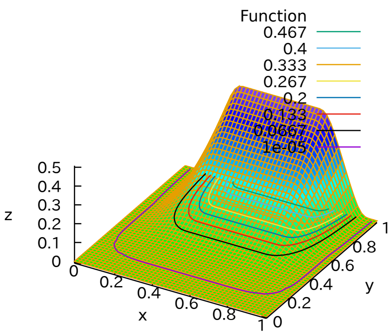
Theorem (Whitney approximation) Let (X, A) be a smooth relative CW complex and $f\colon TX \to TY$ a continuous map from X to a smooth CW complex Y. Suppose f restricts to a smooth map $A \to Y$. Then there exist a smooth map $g\colon X \to Y$ and a continuous homotopy $h\colon TX \times I \to TY$ between f and Tg relative to TA.

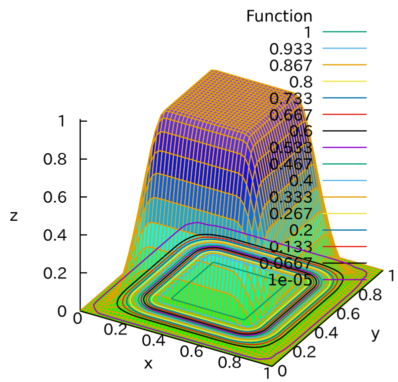
Corollary Let X be a smooth CW complex. Then the natural map $\pi_n(X, x_0) \to \pi_n(TX, x_0)$ is an isomorphism for any $x_0 \in X$ and $n \geq 0$.

Preliminary results

Lemma There exists a smooth deformation retraction $R\colon I^{n+1} \to L^n$, where $L^n = \partial I^n \times I \cup I^n \times \{0\} \subset I^{n+1}$.

Lemma There exists a smooth map $I^n \to I^n$ which restricts to a smooth deformation retraction $I^n - [\epsilon, 1-\epsilon]^n \to \partial I^n$ $(0 < \epsilon < 1/2)$.





Homotopical properties of CW complexes

Proposition (HEP) Let (X, A) be a smooth relative CW complex. Suppose we are given a smooth map $f\colon X \to Y$ and a smooth homotopy $h\colon A \times I \to Y$ satisfying $h_0 = f|A$. Then there exists a smooth homotopy $H\colon X \times I \to Y$ which extends h and satisfies $H_0 = f$.

Proposition (MVP) Let (X, A) be a smooth relative CW complex. For each $n \geq 0$ there exist an open subset $V \subset X^n$ containing X^{n-1} and a smooth map $\rho\colon X^n \to X^n$ such that

(1) $1 \simeq \rho$ rel X^{n-1}

(2) $\rho|V$ gives a deformation retraction $V \to X^{n-1}$.

Expected consequences of MVP

J.H.C. Whitehead's theorem Let $f\colon X \to Y$ be a smooth map between smooth CW complexes. Then the following are equivalent:

(1) f is a homotopy equivalence in $\mathbf{Diff}$

(2) f is a weak homotopy equivalence in $\mathbf{Diff}$

(3) Tf is a weak homotopy equivalence in $\mathbf{Top}$

(4) Tf is a homotopy equivalence in $\mathbf{Top}$

de Rham's theorem If X is a smooth CW complex then

$$I\colon H^n_{\mathsf{dR}}(X, \mathbf{R}) \to H^n(X, \mathbf{R})$$

is an isomorphism for every $n \geq 0$.

Local case

The next proposition is in fact a special case of the theorem, but plays a key role in the proof of the theorem.

Proposition Let f be a continuous map from I^n to a smooth CW complex Y. Then there exists a smooth map $g\colon I^n \to Y$ such that Tg is homotopic to f. If f is already smooth on a cubical subcomplex L of I^n then the homotopy can be taken to be relative to L.

Remark The corollary is an immediate consequence of this.

Sketch of the proof Since $f(I^n)$ is compact, we may assume Y is a finite complex. The proof is by induction on the least integer $m \geq 0$ such that $f(I^n)$ is contained in Y^m of Y.

Let $\{e_1, \ldots, e_r\}$ be the set of m-cells of Y and let $U = \bigcup_{j=1}^{r} \Phi_j(\mathrm{Int}\, I^m)$, where $\Phi_j \colon I^m \to Y$ is the characteristic map for e_j. Then there is an open cover $\{U, V\}$ of Y^m enjoying the following properties:

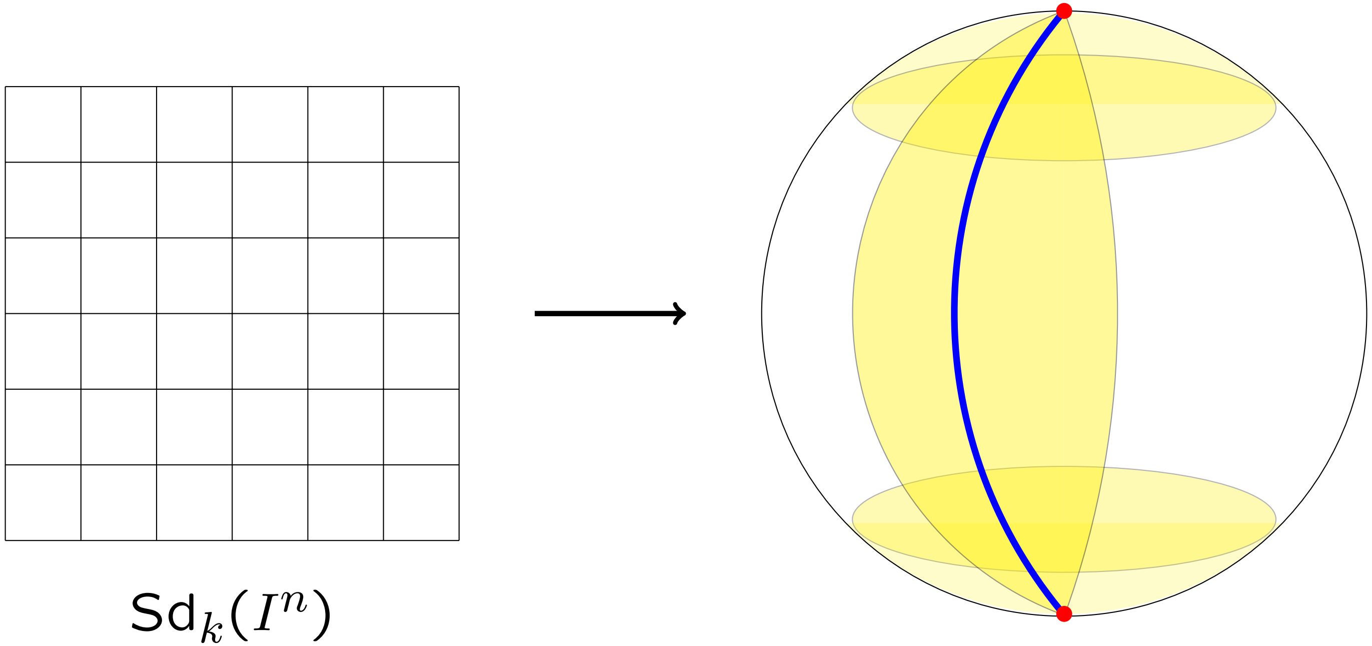
(1) U has a finite number of path components diffeomorphic to $\mathbf{R}^m$.

(2) There is a smooth homotopy $1 \simeq \rho \colon Y^m \to Y^m$ rel Y^{m-1} such that ρ restricts to a retraction $V \to Y^{m-1}$.

Let $\mathrm{Sd}_k(I^n)$ be the cubical subdivision of I^n consisting of subcubes

$$K_J = \left[\tfrac{j_1-1}{k}, \tfrac{j_1}{k}\right] \times \cdots \times \left[\tfrac{j_n-1}{k}, \tfrac{j_n}{k}\right]$$

where $J = (j_1, \cdots, j_n) \in \{1, \cdots, k\}^n$. By taking k large enough, we may assume each $f(K_J)$ is contained in either U or V.



- If $f(K_J) \subset U$ then use the (original) Whitney approximation to construct $f|K_J \simeq g_J$ such that g_J is smooth.
- If $f(K_J) \subset V$ then $\rho(f(K_J)) \subset Y^{m-1}$, so that we can construct $\rho \circ f|K_J \simeq g_J$ such that g_J is smooth by the inductive assumption.

General case

Starting from the trivial homotopy of $g_{-1} = f|A$, we inductively construct a smooth map $g_n\colon X^n \to Y$ and a homotopy $h_n\colon TX^n \times I \to TY$ giving $f|TX^n \simeq Tg_n$ rel TA. The desired map $g\colon X \to Y$ and homotopy $f \simeq Tg$ are obtained by taking their colimits.

Suppose g_{n-1} and h_{n-1} exist. Let $(\Phi_\lambda, \phi_\lambda)\colon (I^n, \partial I^n) \to (X^n, X^{n-1})$ be the characteristic map for the n-cell e_λ ($\lambda \in \Lambda_n$), and put

$$k_\lambda = h_{n-1} \circ (\phi_\lambda \times 1) \cup f \circ \Phi_\lambda\colon \partial I^n \times I \cup I^n \times \{0\} \to TY.$$

Then $k_\lambda \circ R\colon I^n \times I \to TY$ coincides with $h_{n-1} \circ (\phi_\lambda \times 1)$ on $\partial I^n \times I$ and with $f \circ \Phi_\lambda$ on $I^n \times \{0\}$.

But then, there exist a smooth map $g_\lambda\colon I^n \to Y$ extending $g_{n-1} \circ \phi_\lambda$, and a homotopy $h'_\lambda\colon I^n \times I \to TY$ giving $k_\lambda \circ R_1 \simeq Tg_\lambda$ rel ∂I^n.

Thus there is a composite homotopy $h_\lambda\colon f \circ \Phi_\lambda \simeq k_\lambda \circ R_1 \simeq Tg_\lambda$.

Here, h_λ must satisfy the following requirements:

- It should be smooth.
- It should restrict to $h_{n-1} \circ (\phi_\lambda \times \{1\}$ on $\partial I^n \times I$.

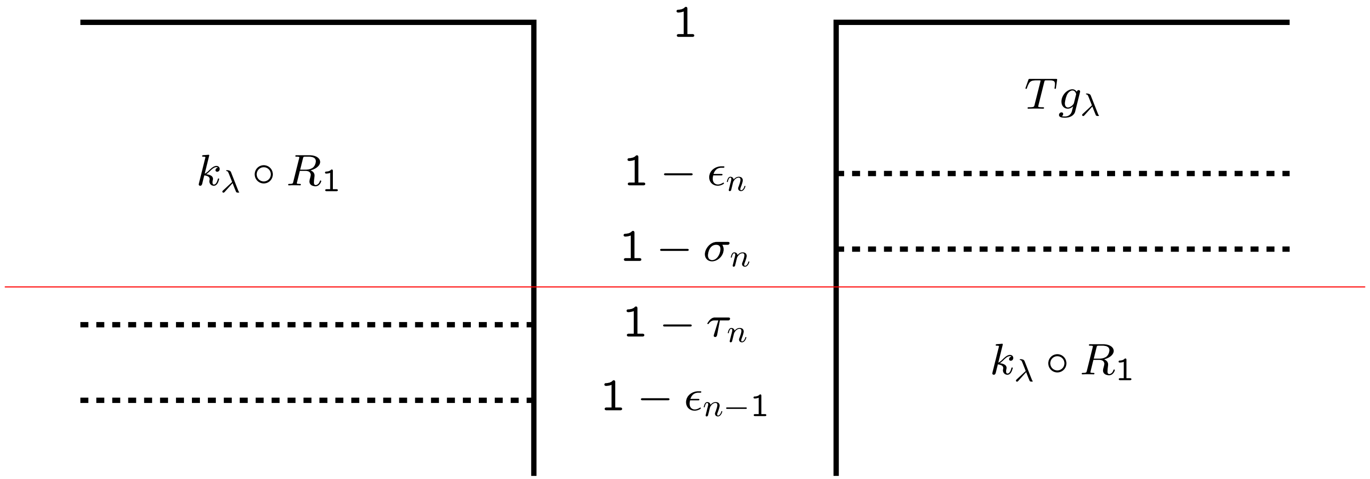
To achieve these, let us choose positive numbers

$$\epsilon_{n-1} > \tau_n > \sigma_n > \epsilon_n \quad (n \geq 1)$$

and suppose h_{n-1} is ϵ_{n-1}-stationary at $X^{n-1} \times \{1\}$.

Reparametrize the homotopies $k_\lambda \circ R$ and h'_λ as follows:

$k_\lambda \circ R \circ (1 \times \theta)$: $h'_\lambda \circ (1 \times \theta')$:



where θ and θ' are non-decreasing functions satisfying

$$\theta(t) = \begin{cases} t, & t \leq 1 - \epsilon_{n-1} \\ 1, & 1 - \tau_n \leq t \end{cases}, \qquad \theta'(t) = \begin{cases} 0, & t \leq 1 - \sigma_n \\ 1, & 1 - \epsilon_n \leq t \end{cases}$$

The resulting homotopy $h_\lambda\colon I^n \times I \to TY$ between $f \circ \Phi_\lambda$ and Tg_λ extends $h_{n-1} \circ (\phi_\lambda \times 1)$ and is ϵ_n-stationary at $I^n \times \{1\}$, i.e. $h_n(s,t) = h_n(s,1)$ holds if $1 - \epsilon_n \leq t \leq 1$.

Now, we have a diagram

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda_n}(TX^{n-1} \times I \coprod I^n \times I) & \xrightarrow{\bigcup_\lambda h_{n-1}\cup h_\lambda} & TY \\ \downarrow & \nearrow & \\ TX^n \times I & & \end{array}$$

Since the vertical arrow is a quotient map, there exists a continuous map $h_n\colon TX^n \times I \to TY$ making the diagram commutative.

Clearly, h_n is ϵ-smashed at $TX^n \times \{1\}$ and gives $f|TX^n \simeq Tg_n$ rel TA, where $g_n\colon X^n \to Y$ is induced by $g_{n-1} \cup g_\lambda\colon X^{n-1} \coprod I^n \to Y$ $(\lambda \in \Lambda_n)$. This completes the proof of **Theorem**.

Appendix

For $0 \leq \epsilon < 1/2$, there exists $\lambda_\epsilon \colon \mathbf{R} \to I$ satisfying:

(1) $\lambda(t) = 0$ for $t \leq \epsilon$

(2) λ is strictly increasing on $[\epsilon, 1-\epsilon]$

(3) $\lambda(t) = 1$ for $1 - \epsilon \leq t$

(4) $\lambda(1-t) = 1 - \lambda(t)$ for all t