Partial quasimorphisms on the Hamiltonian diffeomorphism groups

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Definition of quasimorphism

Definition

Let G be a group. A function $\mu\colon G \to \mathbb{R}$ is called an *quasimorphism on G* if $\exists\ C > 0$ such that $\forall f, \forall g \in G$,

$$|\mu(fg) - \mu(f) - \mu(g)| < C.$$

μ is called *homogeneous* if $\mu(f^n) = n\mu(f)$ holds $\forall f \in G$ and $\forall n \in \mathbb{Z}$.

Example

Let G be a finite group. Then, any function G is a quasimorphism. And, any homogeneous quasimorphism on G is zero function.

History of quasimorphisms 1

Henri Poincaré,
"Mémoire sur les courbes définies par les equations différentialle, J. de Mathématiques 7-8 (1881-2), 375-422, 251-296."
constructed the rotation quasimorphism : $\widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$
(This quasimorphism induces the rotation number : $\mathrm{Homeo}^+(S^1) \to \mathbb{R}/\mathbb{Z}$)

History of quasimorphisms 2

Linear space of quasimorphism has a deep relationship to the 2nd bounded cohomology $H_b^2(G;\mathbb{R})$ of a group G. Since the following Gromov's famous work on bounded cohomology appeared, people has recognized the importance of the bounded cohomology and quasimorphisms.
Mikhael Gromov, "Volume and bounded cohomology", Inst. Hautes Études Sci. Publ. Math. (1982), no. 56, 599 (1983).,

History of quasimorphisms 3

Let M be a closed hyperbolic manifold. Then $\pi_1(M)$ admits a non-trivial homogeneous quasimorphism which is called the de Rham quasimorphism [Barge-Ghys 88].
Epstein and Fujiwara generalized this result. They proved that there are infinitely many non-trivial homogeneous quasimorphisms on a non-elementary word-hyperbolic group. [Epstein-Fujiwara 97]

a conjugation-invariant norm

Definition ([Burago-Ivanov-Polterovich 08])

Let G be a group. A function $\nu\colon G \to \mathbb{R}_{\geq 0}$ is *a conjugation-invariant norm* on G if ν satisfies the following axioms:

(1) $\nu(1) = 0$;

(2) $\nu(f) = \nu(f^{-1})$ for every $f \in G$;

(3) $\nu(fg) \leq \nu(f) + \nu(g)$ for every $f, g \in G$;

(4) $\nu(f) = \nu(gfg^{-1})$ for every $f, g \in G$;

(5) $\nu(f) > 0$ for every $f \neq 1 \in G$.

A function $\nu\colon G \to \mathbb{R}$ is *a conjugation-invariant pseudo-norm* on G if ν satisfies the above axioms (1),(2),(3) and (4).

Example

fragmentation norm, commutator length and Hofer's norm are conjugation-invariant norms.

Stably Unboundedness

Definition ([Burago-Ivanov-Polterovich 08])

For a conjugation-invariant norm ν on a group G, let *$s\nu$* denote the stabilization of ν *i.e.* $s\nu(g) = \lim_{n\to\infty} \frac{\nu(g^n)}{n}$ (this limit exists by Fekete's Lemma). ν is *stably bounded* if $s\nu(g) = 0$ for any $g \in G$. ν is *stably unbounded* otherwise.

Remark

Any group admits a stably bounded norm.

commutator length

Let G be a group. We define a normal subgroup $[G, G]$ of G and a conjugation-invariant norm $\mathrm{cl}\colon [G, G] \to \mathbb{R}$ by

$$[G, G] = \{h \mid \exists f_1, \cdots, f_k, g_1, \cdots, g_k;\ h = [f_1, g_1] \cdots [f_k, g_k]\}.$$

$$\mathrm{cl}(h) = \min\{k \mid \exists f_1, \cdots, f_k, g_1, \cdots, g_k;\ h = [f_1, g_1] \cdots [f_k, g_k]\}.$$

Here $[a, b]$ is the commutator $aba^{-1}b^{-1}$ for $a, b \in G$.

Recall original Bavard's duality theorem

Recall

Proposition

Let μ be a homogeneous quasimorphism on a perfect group G. For $g \in G$ with $\mu(g) \neq 0$, $\mathrm{scl}(g) > 0$.

Theorem (Bavard's duality theorem, [Bavard 91])

Let G be a perfect group. For $g \in G$ with $\mathrm{scl}(g) > 0$, there is a homogeneous quasimorphism μ such that $\mu(g) \neq 0$.

Today's theme

Generalize these relationships to partial quasimorphisms and conjugation-invariant norms.

Fragmententation norm

To define partial quasimorphism, we define

Definition

Let G be a group and H a subset of G. We define the fragmentation norm q_H with respect to H by for an element f of G,

$$q_H(f) = \min\{k; \exists g_1 \ldots, g_k \in G, \exists h_1, \ldots h_k \in H \\ \textit{such that } f = g_1^{-1} h_1 g_1 \cdots g_k^{-1} h_k g_k\}.$$

If there is no such decomposition of f as above, we put $q_H(f) = \infty$. H *c-generates* G if such decomposition as above exists for any $f \in G$.

Partial quasimorphism

Now, we define partial quasimorphism (subset-controlled quasimorphism)!

Definition

Let H, G' be subgroups of a group G. A function $\mu\colon G' \to \mathbb{R}$ is called an *H-quasimorphism on G'* if there exists a positive number C such that for any elements f, g of G',

$$|\mu(fg) - \mu(f) - \mu(g)| < C \cdot \min\{q_H(f), q_H(g)\}.$$

μ is called *homogeneous* if $\mu(f^n) = n\mu(f)$ holds for any element f of G' and any integer n. μ is called *semi-homogeneous* if $\mu(f^n) = n\mu(f)$ holds for any element f of G' and any non-negative integer n.

Quasimorphism is partial quasimorphism

Definition

Let H, G' be subgroups of a group G. A function $\mu\colon G' \to \mathbb{R}$ is called an *H-quasimorphism on G'* if there exists a positive number C such that for any elements f, g of G',

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Remark

Any quasimorphism on a group G is a G-quasimorphism.

Definition of Hamiltonian diffeomorphism group (1)

Let (M, ω) be a symplectic manifold. Define the sympletomorphism group of (M, ω) by

$$\mathrm{Symp}(M, \omega) = \{f \in \mathrm{Diff}_c(M); f^*\omega = \omega\}.$$

Define $\mathrm{Symp}_c(M, \omega)$ as its identity component.

Definition of Hamiltonian diffeomorphism group (2)

Define the flux homomorphism $\mathrm{Flux}\colon \widetilde{\mathrm{Symp}}_c(M,\omega) \to H^1_c(M;\mathbb{R})$ by

$$\mathrm{Flux}([\tilde{f}]) = \int_0^1 [\iota_{X_t}\omega],$$

where X_t is the (time-dependent) vector field which generates an isotopy $\{f^t\}_{t\in[0,1]}$ representing $\tilde{f} \in \widetilde{\mathrm{Symp}}_c(M,\omega)$.

Definition of Hamiltonian diffeomorphism group (3)

Define the flux group $\Gamma = \mathrm{Flux}(\pi_1(\mathrm{Symp}_c(M,\omega)))$. Then we define the group of Hamiltonian diffeomorphisms by

$$\mathrm{Ham}(M,\omega) = (\mathrm{Flux})^{-1}(\Gamma).$$

Remark

By the flux conjecture (conjectured by Banyaga, proved by Ono), Γ is known to be a discete subgroup of $H^1_c(M;\mathbb{R})$ in many cases.

Definition of Hamiltonian diffeomorphism group (4)

Remark

For simplicity, $\mathrm{Ham}(M,\omega)$ is often denoted by $\mathrm{Ham}(M)$.

Remark

Since $H_c^1(\mathbb{R}^{2n};\mathbb{R}) = 0$, $\mathrm{Ham}(\mathbb{R}^{2n},\omega_0) = \mathrm{Symp}_0(\mathbb{R}^{2n},\omega_0)$.

History of partial quasimorphisms

- Viterbo constructed a spectral invariant $c\colon \mathrm{Ham}(T^*)N \to \mathbb{R}$ [Viterbo 92]. Here, N is a closed manifold.
- Schwarz and Oh constructed spectral invariants $c_a\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ by using the Hamiltonian Floer theory. Here M is a symplectic manifold and a is an element of the quantum homology $QH_*(M)$ [Schwarz 00], [Oh 02], [Oh 06] *et al.*
- Entov and Polterovich proved that c_a is a quasimorphism under the assumption of semi-simplicity of $QH_*(M)$ [Entov-Polterovich 03].
- Entov and Polterovich proved that c_a is a partial quasimorphism with respect to displaceable subsets when $a = [M] \in QH_*(M)$ [Entov-Polterovich 06]. This is the first time when a concept of partial quasimorphism appeared.

After Entov-Polteorvich's work 1

Appilication of (asymptotic) Oh-Schwarz spectral invariant

1. Entov and Polterovich gave concepts of heaviness and superheaviness of closed subsets of symplectic manifolds. Heavy or superheavy subsets are known to be non-displaceable.
2. Buhovsky, Entov and Polterovich considered the Poisson bracket invariant pb_4. They gave a lower bound of pb_4 by using superheavy subsets. pb_4 is also useful for founding a Hamiltonian chord between two disjoint subsets.
3. Biran, Polterovich and Salamon defined a relative symplectic capacity which measures the existence of non-contractible trajectories of Hamiltonian isotopies. K. gave an upper bound of Biran, Polterovich and Salamon's capacity by using an asymptotic Oh-Schwarz's spectral invariant.

After Entov-Polteorvich's work 2

Generalization of Entov-Polterovich's work.

1. Monzner-Vichery-Zapolsky: cotangent bundle.
2. K, *et. al.*: using Lagrangian spectral invariant (in progress).
3. Borman-Zapolsky: constructed a partial quasimorphism contactomorphism group (using Givental's quasimorphism).
4. Borman: constructed a partial quasimorphism by "reduction".
5. Fukaya-Oh-Ohta-Ono: constructed a partial quasimorphism by bulk-deformed Hamiltonian Floer theory.
6. Le Roux-Humiliere-Seyfaddini: On surface case, give another description by "more classical theory in surface dynamics".

Burago-Ivanov-Polterovich Problem

Problem ([Burago-Ivanov-Polterovich 08]'s Problem)

Does there exist a group G such that

(1) *G is perfect i.e. $G = [G, G]$;*

(2) *The commutator length of G is stably bounded;*

(3) *G admits a stably unbounded conjugation-invariant norm?*

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(1) *G is perfect i.e. $G = [G, G]$;*

(2) *The commutator length of G is stably bounded;*

(3) *G admits a stably unbounded conjugation-invariant norm?*

Answer ([K.-16, JSG], [Brandenbursky-Kedra15], [Kimura 17]): Yes, there is!

What are examples?

Let G be the group $\mathrm{Ham}(\mathbb{R}^{2n})$ of Hamiltonian diffeomorphisms or the infinite braid group $B_\infty = \bigcup_i B_i$.
The following was already known.

Proposition ([Banyaga 78] *et. al.*)

Then the commutator subgroup $[G, G]$ is perfect and the commutator length is stably bounded on $[G, G]$.

The following is our new observation.

Theorem

$[G, G]$ admits a stably unbounded conjugation-invariant norm (when $G = \mathrm{Ham}(\mathbb{R}^{2n})$, [K.-16, JSG], when $G = B_\infty$, [Brandenbursky-Kedra15] and [Kimura 17]).

Subset-controlled commutator length

Let G be a group and H a subgroup of G and $p, q \in \mathbb{Z}_{>0} \cup \{\infty\}$. We define a normal subgroup $[G, G]^H_{p,q}$ of G and a conjugation-invariant norm $\mathrm{cl}^H_{p,q} \colon [G, G]^H_{p,q} \to \mathbb{R}$ by

$$\begin{aligned}[G, G]^H_{p,q} =&\{h \mid \exists f_1, \cdots, f_k, g_1, \cdots, g_k;\\ &\quad q_H(f_i) \leq p, q_H(g_j) \leq q\,(i, j = 1, \cdots, k);\ h = [f_1, g_1] \cdots [f_k, g_k]\}.\end{aligned}$$

$$\begin{aligned}\mathrm{cl}^H_{p,q}(h) =& \min\{k \mid \exists f_1, \cdots, f_k, g_1, \cdots, g_k;\\ &\quad q_H(f_i) \leq p, q_H(g_j) \leq q\,(i, j = 1, \cdots, k);\ h = [f_1, g_1] \cdots [f_k, g_k]\}.\end{aligned}$$

Here $[a, b]$ is the commutator $aba^{-1}b^{-1}$ for $a, b \in G$.

Propositions

Proposition ([K.-16, JSG],[Kimura 17])

Let (G, H) *be* $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ *or* (B_∞, B_n). *Then* $[G, G]^H_{p,q} = [G, G]$ *holds for any* $p, q \in \mathbb{Z}_{>0} \cup \{\infty\}$.

Proposition ([K.-16, JSG])

If there exists a semi-homogeneous H*-quasimorphism* μ *on* $[G, G]^H_{p,q}$ *with* $\mu(g) \neq 0$ *for some* $g \in [G, G]^H_{p,q}$, *then* $scl^H_{p,q}(g) > 0$ *holds for any* $p, q \in \mathbb{Z}_{>0} \cup \{\infty\}$.

sufficient to prove

Thus it is sufficient to prove

Theorem ([K.-16, JSG], [Kimura 17])

Let (G, H) *be* $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ *or* (B_∞, B_n).
There exists a non-trivial homogeneous H-quasimorphism $\mu\colon G \to \mathbb{R}$.

Kimura's construction

We define $\sigma\colon B_\infty \to \mathbb{Z}$ by

$$\sigma(b) = (\text{the signature of } \bar{b}),$$

where $\bar{\sigma}$ is the braid closure of b.

$$\bar{\sigma}(b) = \lim_{k\to\infty} \frac{\sigma(b^k)}{k}$$

Theorem ([Kimura 17])

$\bar{\sigma}$ is a homogeneous B_i-quasimorphism.

Theorem ([Gambaudo-Ghys 04])

$\bar{\sigma}|_{B_i}$ is a homogeneous quasimorphism and $\bar{\sigma}(b) \neq 0$ for some $b \in [B_i, B_i]$.

Construction on $\mathrm{Ham}(\mathbb{R}^{2n})$ step 1

For $g \in \mathrm{Ham}(\mathbb{R}^{2n})$, choose an isotopy $(g^t)_{t\in[0,1]}$ in $\mathrm{Ham}(\mathbb{R}^{2n})$ between $g^0 = 1$ and $g^1 = g$.
For each point x in $\mathbb{R}^{2n}$, the differential $(dg^t)\colon T_x\mathbb{R}^{2n} \to T_{g^t(x)}\mathbb{R}^{2n}$ is given as a $2n \times 2n$ matrix $A(x, g^t) \in \mathrm{Sp}(2n, \mathbb{R})$.
Thus the path $(A(x, g^t))_{t\in[0,1]}$ on $\mathrm{Sp}(2n, \mathbb{R})$ represents an element of the universal covering $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$.
Let $\tilde{\beta}\colon \widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \to \mathbb{R}$ be the Maslov quasimorphism.

Construction on $\mathrm{Ham}(\mathbb{R}^{2n})$ step 2

$\tilde{\beta}([(A(x,g^t))_{t\in[0,1]}])$ does not depend on the choice of an isotopy $(g_t)_{t\in[0,1]}$, and we denote $\tilde{\beta}([(A(x,g^t))_{t\in[0,1]}])$ by $\beta(g,x)$.
We define the function $\mathcal{B}\colon \mathrm{Ham}(\mathbb{R}^{2n}) \to \mathbb{R}$ by

$$\mathcal{B}(g) = \int_{x\in\mathbb{R}^{2n}} \beta(g,x)\omega_0^n.$$

Note that since $\beta(g,x) = 0$ for $x \notin \bigcup_{t\in[0,1]}(g^t)^{-1}(\mathrm{supp}(g^t))$, $\mathcal{B}(g) < \infty$ for any $g \in \mathrm{Ham}(\mathbb{R}^{2n})$.

$$\bar{\mathcal{B}}(g) = \lim_{k\to\infty} \frac{\mathcal{B}(g^k)}{k}.$$

K. prove that $\bar{\mathcal{B}}$ is a non-trivial homogeneous $\mathrm{Ham}(\mathbb{B}^{2n})$-quasimorphism.

Barge-Ghys construction

Theorem ([Barge-Ghys 92])

$\bar{\mathcal{B}}|_{\mathrm{Ham}(\mathbb{B}^{2n})}$ is a homogeneous quasimorphism on $\mathrm{Ham}(\mathbb{B}^{2n})$ and $\bar{\mathcal{B}}(f) \neq 0$ for some $f \in [\mathrm{Ham}(\mathbb{B}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n})]$.

Remark

$\mathrm{Ham}(\mathbb{R}^{2n})$ does not admit a non-trivial homogeneous quasimorphism.

Problem on MCG

Problem

Is there a non-trivail semi-homogeneous $\mathrm{MCG}(\Sigma_i^1)$-quasimorphism on $\mathrm{MCG}(\Sigma_\infty) = \bigcup_i \mathrm{MCG}(\Sigma_i^1)$?

Theorem ([Endo-Kotschick 01])

There is a non-trivial homogeneous quasimorphism on $\mathrm{MCG}(\Sigma_i^1)$ if $i \geq 2$.

Remark

$\mathrm{MCG}(\Sigma_\infty)$ does not admit a non-trivial homogeneous quasimorphism.

Recall original Bavard's duality theorem

Recall

Proposition

Let μ be a homogeneous quasimorphism on a perfect group G. For $g \in G$ with $\mu(g) \neq 0$, $\mathrm{scl}(g) > 0$.

Theorem (Bavard's duality theorem, [Bavard 91])

Let G be a perfect group. For $g \in G$ with $\mathrm{scl}(g) > 0$, there is a homogeneous quasimorphism μ such that $\mu(g) \neq 0$.

Bavard's duality theorem on conjugation-invariant norm

Recall

Proposition ([K.-16, JSG])

If there exists a semi-homogeneous H-quasimorphism μ on $[G, G]^H_{p,q}$ with $\mu(g) \neq 0$ for some $g \in [G, G]^H_{p,q}$, then $scl^H_{p,q}(g) > 0$ holds for any $p, q \in \mathbb{Z}_{>0} \cup \{\infty\}$.

We have the following Bavard-type duality theorem.

Theorem ([K.-17, PJM])

Let (G, H) be $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ or (B_∞, B_n) or $(\mathrm{MCG}(\Sigma_\infty), \mathrm{MCG}(\Sigma^1_l))$ and ν a conjugation-invariant pseudo-norm on G. Then, for any element g of G such that $s\nu(g) > 0$, there exists a homogeneous H-quasimorphism $\mu \colon G \to \mathbb{R}$ such that $\mu(g) \neq 0$.

Construction of μ

Theorem ([K.-17, PJM])

Let (G, H) be $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ or (B_∞, B_n) or $(\mathrm{MCG}(\Sigma_\infty), \mathrm{MCG}(\Sigma_l^1))$ and ν a conjugation-invariant pseudo-norm on G. Then, for any element g of G such that $s\nu(g) > 0$, there exists a homogeneous H-quasimorphism $\mu\colon G \to \mathbb{R}$ such that $\mu(g) \neq 0$.

Remark

Construction of μ is very far from concrete. We have to use the Hahn-Banach theorem (the axom of choice)! (original Bavard's duality theorem also uses the Hahn-Banach theorem)

Extension problem of pre-quasimorphisms

Now, we give an extrinsic application of our Bavard-type duality theorem.

Conjecture ([K.-17, PAMSB])

Let (G, H) *be* $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ *or* (B_∞, B_n). *For a semi-homogeneous* H*-quasimorphism* μ *on* $[G, G]$*, there exists a homogeneous* H*-quasimorphism* $\hat{\mu}$ *on* G *such that* $\hat{\mu}|_{[G,G]} = \mu$. *In particular, any semi-homogeneous* H*-quasimorphism on* $[G, G]$ *is a homogeneous* H*-quasimorphism.*

Supporting Theorem

Our main theorem is the following one which supports the above conjecture.

Theorem ([K.-17, PAMSB])

Let (G, H) *be* $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ *or* (B_∞, B_n). *For a semi-homogeneous* H*-quasimorphism* μ *on* $[G, G]$ *and an element* g *of* $[G, G]$ *such that* $\mu(g) \neq 0$, *there exists a homogeneous* H*-quasimorphism* $\hat{\mu}_g$ *on* G *such that* $\hat{\mu}_g(g) \neq 0$.

Kimura's proposition

Let G, H denote B_∞, B_n, respectively.
Let σ_1 denote the first standard Artin generator of B_∞. It is known that $\{\sigma_1^{\pm 1}\}$ c-generates G.

Proposition ([Kimura 17])

The restriction of $q_{\{\sigma_1^{\pm 1}\}}$ to $[G, G]$ is quasi-isometric to $cl_{p,q}^H$ i.e. $\exists C > 0$ such that $C^{-1} \cdot cl_{p,q}^H(g) \leq q_{\{\sigma_1^{\pm 1}\}}(g) \leq C \cdot cl_{p,q}^H(g)$ holds for any $g \in [G, G]$.

Recall Theorem

Recall:

Theorem ([K.-17, PAMSB])

Let (G, H) *be* $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ *or* (B_∞, B_n)*. For a semi-homogeneous* H*-quasimorphism* μ *on* $[G, G]$ *and an element* g *of* $[G, G]$ *such that* $\mu(g) \neq 0$*, there exists a homogeneous* H*-quasimorphism* $\hat{\mu}_g$ *on* G *such that* $\hat{\mu}_g(g) \neq 0$*.*

Proof of Theorem

Proof of Theorem when $(G,H) = (B_\infty, B_n)$.

Let g be an element of $[G,G]$ and μ a semi-homogeneous H-quasimorphism on $[G,G]$ with $\mu(g) \neq 0$. Since $\mu(g) \neq 0$, Proposition implies $scl^H_{p,q}(g) > 0$. Since $q_{\{\sigma_1^{\pm 1}\}}|_{[G,G]}$ is quasi-isometric to $cl^H_{p,q}$, $sq_{\{\sigma_1^{\pm 1}\}}(g) > 0$. Then our Bavard-type duality theorem implies that there exists a homogeneous H-quasimorphism $\hat{\mu}_g$ on G such that $\hat{\mu}_g(g) \neq 0$. □

Hamiltonian analogue of $q_{\{\sigma_1^{\pm 1}\}}$

Let G, H denote $\mathrm{Ham}(\mathbb{R}^{2n})$, $\mathrm{Ham}(\mathbb{B}^{2n})$, respectively.
For proving Theorem when $(G, H) = (\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$, it is important to construct a Hamiltonian analogue of $q_{\{\sigma_1^{\pm 1}\}}$.
Let $F \colon \mathbb{R}^{2n} \to \mathbb{R}$ be a (time-independent) Hamiltonian function such that $\phi_F^1 \notin [G, G]$ and h an element of $[G, G]$.
We define the conjugation-invariant norm $\nu_{F,h}$ by

$$\nu_{F,h} = q_{\{\phi_F^t\}_{t\in[-1,1]} \cup \{h^{\pm 1}\}}.$$

The subset $\{\phi_F^t\}_{t\in[-1,1]} \cup \{h^{\pm 1}\}$ c-generates G and thus $\nu_{F,h}$ is a conjugation-invariant norm on G.

A Hamiltonian analogue of Kimura's Proposition

We use the following proposition which is a Hamiltonian analogue of Kimura's Proposition.

Proposition ([K.-17, PAMSB])

The restriction of $\nu_{F,h}$ to $[G, G]$ is quasi-isometric to $cl^H_{p,q}$.

Then, we can prove the theorem similarly to the braid case.

Displaceability in terms of group theory

Definition

Let G be a group, H a subgroup of G of G. We define the set $\mathrm{D}(H)$ of maps *displacing* H by

$$\mathrm{D}(H) = \{\phi \in G; (\phi)^{-1}H\phi \text{ commutes with } H\}.$$

Example

Put $G = \mathrm{Ham}(M)$
Let U be an open subset of M and put $H = \mathrm{Ham}(U)$.
Then, since $(\phi)^{-1}H\phi = \mathrm{Ham}(\phi(U))$,

$$\mathrm{D}(H) = \{\phi \in G; \phi(U) \cap U = \emptyset\}$$

Property FD

Definition

Let G be a group and H a subgroup of G. (G, H) satisfies the property FD if G and H satisfies the following conditions.

(1) G is c-generated by H,

(2) $\mathrm{D}(H) \neq \emptyset$.

A group G satisfies the property FD if (G, H) satisfies the property FD for some subgroup H.

For a group G which satisfies the property FD, we define the set $\mathrm{FD}(G)$ by

$$\mathrm{FD}(G) = \{H \leq G; (G, H) \text{ satisfies the property FD}\}.$$

Bavard-type duality theorem (conjecture)

Conjecture

Let G be a group satisfying the property FD *and ν a conjugation-invariant pseudo-norm on G. Then, for any element g of G such that $s\nu(g) > 0$, there exists a function $\mu\colon G \to \mathbb{R}$ which is a semi-homogeneous H-quasimorphism for any element H of* $\mathrm{FD}(G)$ *such that $\mu(g) > 0$.*

In the present section, let G, H denote $\mathrm{Ham}(\mathbb{R}^{2n})$, $\mathrm{Ham}(\mathbb{B}^{2n})$, respectively. We follow the notion of [E] and thus let ϕ_F^t denote the time-t map of the Hamiltonian flow generated by F for a (time-dependent) Hamiltonian function $F\colon \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$.

Definition ([C],[Banyaga 78])

The Calabi homomorphism $\mathrm{Cal}\colon \mathrm{Ham}(\mathbb{R}^{2n}) \to \mathbb{R}$ is defined by

$$\mathrm{Cal}(h) = \int_0^1 \int_M H{\omega_0}^n dt \text{ for a Hamiltonian diffeomorphism } h,$$

where $H\colon \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ is a Hamiltonian function which generates h. $\mathrm{Cal}(h)$ does not depend on the choice of generating Hamiltonian function H and thus the functional Cal is a well-defined homomorphism.

For proving Theorem 4.4 when $(G, H) = (\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$, it is important to construct a Hamiltonian analogue of $q_{\{\sigma_1^{\pm 1}\}}$. Let $F\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a (time-independent) Hamiltonian function such that $\phi_F^1 \notin \mathrm{Ker}(\mathrm{Cal})$ and h an element of $\mathrm{Ker}(\mathrm{Cal})$. Note that $\mathrm{Cal}(\phi_F^t) = t\mathrm{Cal}(\phi_F^1)$. We define the conjugation-invariant norm $\nu_{F,h}$ by $\nu_{F,h} = q_{\{\phi_F^t\}_{t\in\mathbb{R}} \cup \{h^{\pm 1}\}}$. Since $[G, G]$ is a simple group and $[G, G] = \mathrm{Ker}(\mathrm{Cal})$ ([Banyaga 78]), the subset $\{\phi_F^t\}_{t\in\mathbb{R}} \cup \{h^{\pm 1}\}$ c-generates G. Thus $\nu_{F,h}$ is a conjugation-invariant norm on G.

Proposition

The restriction of $\nu_{F,h}$ to $[G, G]$ is G-extremal.

To prove Proposition 5.2, we use the following lemma.

Lemma

Let ν be a G-invariant norm on $[G, G]$. There exists a positive constant $C_{F,\nu}$ which depends only on F and ν such that $\nu([g, \phi_F^t]) < C_{F,\nu}$ holds for any element g of G.

Proof.

Let R be a sufficient large number such that $\mathrm{Supp}(F) \subset Q_R$ where $Q_R = [-R, R]^{2n} \subset \mathbb{R}^{2n}$. Let h_0 be an element of $[G, G]$ such that $Q_R \cap h_0(Q_R) = \emptyset$. Note that $\nu(h_0)$ depends only on F and ν. Fix an element g of G and take an element h_g of G such that $h_g(Q_R) = Q_R$ and $h_g h_0(Q_R) \cap (Q_R \cup \mathrm{Supp}(g)) = \emptyset$. Then $(h_g h_0 h_g^{-1})(\phi_F^t)^{-1}(h_g h_0 h_g^{-1})^{-1}$ commutes with ϕ_F^t and g and thus $[g, \phi_F^t] = [g, [\phi_F^t, h_g h_0 h_g^{-1}]]$. Since ν is a G-invariant norm on $[G, G]$,

$$\begin{aligned}
\nu([g, \phi_F^t]) &\leq \nu(g[\phi_F^t, h_g h_0 h_g^{-1}]g^{-1}) + \nu([\phi_F^t, h_g h_0 h_g^{-1}]^{-1}) \\
&= 2\nu([\phi_F^t, h_g h_0 h_g^{-1}]) \\
&\leq 2(\nu(\phi_F^t(h_g h_0 h_g^{-1})(\phi_F^t)^{-1}) + \nu((h_g h_0 h_g^{-1})^{-1})) \\
&= 4\nu(h_g h_0 h_g^{-1}) = 4\nu(h_0).
\end{aligned}$$

$\square$

Proof of Proposition 5.2 (1)

Let ϕ be an element of $[G, G]$ and m a natural number such that $\nu_{F,h}(\phi) \leq m$. Then, by the definition of $\nu_{F,h}$, there exist $f_1, \ldots, f_m \in \{\phi_F^t\}_{t\in\mathbb{R}} \cup \{h^{\pm 1}\}$ and $g_1, \ldots, g_m \in G$ such that $\phi = g_1^{-1} f_1 g_1 \cdots g_m^{-1} f_m g_m$. We define a function $\tau\colon \{\phi_F^t\}_{t\in\mathbb{R}} \cup \{h^{\pm 1}\} \to \mathbb{R}$ by

$$\tau(f) = \begin{cases} t & (\text{If } f = \phi_F^t), \\ 0 & (\text{If } f \in \{h^{\pm 1}\}). \end{cases}$$

We define real numbers T_k $(k = 1, \ldots, m+1)$ by $T_k = \sum_{i=1}^{k-1} \tau(f_i)$ and set $T_1 = 0$. Then we define elements α_k $(k = 1, \ldots, m)$ of $\mathrm{Ker}(\mathrm{Cal}) = [G, G]$ by

$$\alpha_k = \begin{cases} [\phi_F^{T_k} g_k^{-1}, \phi_F^{t_k}] & (\text{If } f_k = \phi_F^{t_k}), \\ (\phi_F^{T_k} g_k^{-1}) f_k (\phi_F^{T_k} g_k^{-1})^{-1} & (\text{If } f_k \in \{h^{\pm 1}\}). \end{cases}$$

Proof of Proposition 5.2 (2)

Fix a G-invariant norm ν on $[G, G]$. Note that Lemma 5.3 implies $\nu(\alpha_k) \leq \max\{C_{F,\nu}, \nu(h)\}$ holds for any k. Since $\phi_F^{T_k} g_k^{-1} f_k g_k = \alpha_k \phi_F^{T_{k+1}}$ holds for any k,

$$\begin{aligned}\phi &= \phi_F^{T_1} g_1^{-1} f_1 g_1 \cdots g_m^{-1} f_m g_m = \alpha_1 \phi_F^{T_2} g_2^{-1} f_2 g_2 \cdots g_m^{-1} f_m g_m \\ &= \alpha_1 \alpha_2 \phi_F^{T_3} g_3^{-1} f_3 g_3 \cdots g_m^{-1} f_m g_m = \ldots = \alpha_1 \cdots \alpha_m \phi_F^{T_{m+1}},\end{aligned}$$

holds. Since $\phi \in \mathrm{Ker}(\mathrm{Cal})$ and $\alpha_k \in \mathrm{Ker}(\mathrm{Cal})$ for any k, $T_{m+1} = 0$ and thus $\phi = \alpha_1 \cdots \alpha_m$ holds. Since $\nu(\alpha_k) \leq \max\{C_{F,\nu}, \nu(h)\}$ holds for any k, $\nu(\phi) \leq \max\{C_{F,\nu}, \nu(h)\} \cdot m$ holds. Hence
$\nu(\phi) \leq \max\{C_{F,\nu}, \nu(h)\} \cdot \nu_{F,h}(\phi)$ holds for any element ϕ of $[G, G]$.

The proof of Theorem 4.4 when $(G, H) = (\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ is completely similar to the one when $(G, H) = (B_\infty, B_n)$ if we replace Proposition 4.3 by Proposition 5.2.

Symplectic Rigidity

Roughly speaking, Symplectic Rigidity in symplectic topology means proving some "rigidity phenomena" by using some "obstructive invariant".

Example

Proving non-displaceablity of a subset of a symplectic manifold by using Oh-Schwarz spectral invariant or Lagrangian Floer theory.

Symplectic Flexibility

Recall: Symplectic Rigidity in symplectic topology means proving some "rigidity phenomena" by using some "obstructive invariant".

Now:

Symplectic Flexibility in symplectic topology means proving the existence of "obstructive invariant" if "rigidity phenomena" holds.

Now, we explain

1. Pose a Bavard-type duality theorem (this is still now a conjecture) purely written in terms of group theory.
2. The above Bavard-tyoe duality theorem implies a "symplectic flexibility" statement.

Displaceability in symplectic topology

Let (M, ω) be a closed symplectic manifold. A subset X of (M, ω) is *displaceable* if $\bar{X} \cap \phi(X) = \emptyset$ for some Hamiltonian diffeomorphism ϕ. X is *non-displaceable* otherwise. Let $\mathrm{DO}(M)$ denote the set of displaceable open subsets of (M, ω). A subset X of a symplectic manifold M is *stably displaceable* if $X \times S^1$ is displaceable in $M \times T^*S^1$. X is *stably non-displaceable* otherwise.

Remark

A stably displaceable subset is displaceable. Thus, a stably non-displaceable subset is non-displaceable.

Displaceability in terms of group theory

Definition

Let G be a group, H a subgroup of G of G. We define the set $\mathrm{D}(H)$ of maps *displacing* H by

$$\mathrm{D}(H) = \{\phi \in G; (\phi)^{-1} H \phi \text{ commutes with } H\}.$$

Example

Put $G = \mathrm{Ham}(M)$
Let U be an open subset of M and put $H = \mathrm{Ham}(U)$.
Then, since $(\phi)^{-1} H \phi = \mathrm{Ham}(\phi(U))$,

$$\mathrm{D}(H) = \{\phi \in G; \phi(U) \cap U = \emptyset\}$$

Property FD

Definition

Let G be a group and H a subgroup of G. (G, H) satisfies the property FD if G and H satisfies the following conditions.

(1) G is c-generated by H,

(2) $\mathrm{D}(H) \neq \emptyset$.

A group G satisfies the property FD if (G, H) satisfies the property FD for some subgroup H.

For a group G which satisfies the property FD, we define the set $\mathrm{FD}(G)$ by

$$\mathrm{FD}(G) = \{H \leq G; (G, H) \text{ satisfies the property FD}\}.$$

Bavard-type duality theorem (conjecture)

Conjecture

Let G be a group satisfying the property FD *and ν a conjugation-invariant pseudo-norm on G. Then, for any element g of G such that $s\nu(g) > 0$, there exists a function $\mu\colon G \to \mathbb{R}$ which is a semi-homogeneous H-quasimorphism for any element H of* $\mathrm{FD}(G)$ *such that $\mu(g) > 0$.*

Heaviness (Recall)

Now, remember the definition of heaviness.

Definition ([Entov-Polterovich 09])

Let (M, ω) be a closed symplectic manifold and a an idempotent of $QH_*(M, \omega)$. A compact subset X of (M, ω) is *a-heavy* if for any normalized Hamiltonian function $F\colon S^1 \times M \to \mathbb{R}$,

$$-\mu_a(\phi_F) \geq \mathrm{vol}(M) \cdot \inf_{S^1 \times X} F,$$

where $\mathrm{vol}(M) = \int_M \omega^m$.

In particular, if X is a-heavy, $\mu_a(\phi_F) < 0$ for any normalized Hamiltonian function F with $F|_{S^1 \times X} > 0$.

Also recall that μ_a is a semi-homogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any $U \in \mathrm{DO}(M)$.

From Bavard-type duality to symplectic flexibility

Proposition

Assume that the above conjecture holds.
Let X be a stably non-displaceable compact subset of a closed symplectic manifold (M, ω). For any normalized Hamiltonian function $F\colon S^1 \times M \to \mathbb{R}$ with $F|_{S^1 \times X} > 0$, there exists a function $\mu_F\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ which is a semi-homogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathrm{DO}(M)$ such that $\mu_F(\phi_F) < 0$.

This Proposition states that "stably non-displaceable subsets are heavy" (symplectic flexibility!) in a very rough sense if the above conjecture holds.

Polterovich's theorem

To prove Proposition, we use the following Polterovich's theorem.

Theorem ([Polterovich 98])

Let X be a stably non-displaceable subset of a closed symplectic manifold (M, ω). For any normalized Hamiltonian function $F\colon S^1 \times M \to \mathbb{R}$ with $F|_{S^1 \times X} \geq p$ for some positive number p, $||\phi_F|| \geq p$.
Here $||\cdot||\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}_{\geq 0}$ is Hofer's norm which is known to be a conjugation-invariant pseudo-norm.

Proof of Proposition

Proof of Proposition.

Since X is compact, there exists some positive number p with $F|_{S^1\times X} \geq p$. For any positive integer n, we define a Hamiltonian function $F^{(n)}\colon S^1 \times M \to \mathbb{R}$ by $F^{(n)}(t,x) = n \cdot F(nt,x)$. Note that $\phi_{F^{(n)}} = (\phi_F)^n$. Then, by $F^{(n)}|_{S^1\times X} \geq np$ and Polterovich's Theorem, $||(\phi_F)^n||_{\geq} np$ for any positive integer n. Since $\widetilde{\mathrm{Ham}}_U(M) \in \mathsf{FD}(\widetilde{\mathrm{Ham}}(M))$ for any element U of $\mathsf{DO}(M)$, by our conjecure, there exists a function $\mu'_F\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ which is a semi-homogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathsf{DO}(M)$ such that $\mu'_F(\phi_F) > 0$. Then $-\mu'_F$ is a desired function. □

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